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Some Steady State Exact Solutions Of The Wind-Driven Circulation Equations For A Rigid-Lid Shallow Lake

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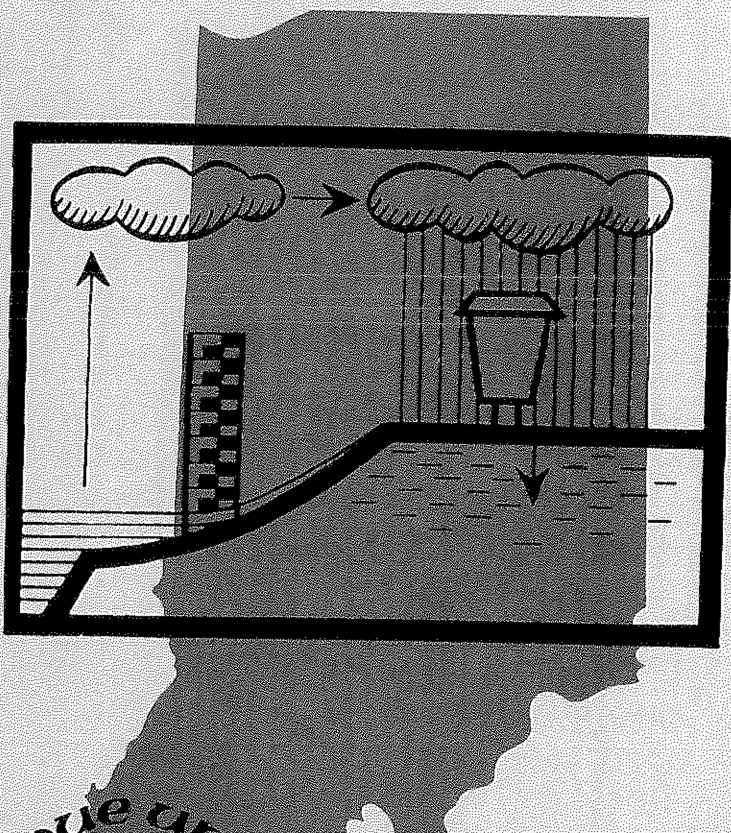

by

Aldo Giorgini

June 1979

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ABSTRACT

This study presents a method for solving the lake circulation equations under the rigid-lid approximation and for small shallow lakes. The driving force, simulating the wind, is constituted by a one-directional she r whose intensity may be variable across the lake.

The method relaxes the bathymetry as a datum and finds it instead, together with the flow field, by suitable constraints on the flow field itself.

The most interesting case of uniform surface shear (uniform wind intensity) is studied for the following subcases:

a) a lake;

b) an underwater hill in an otherwise constant-depth water body of infinite horizontal extent;

c) an underwater pool in an otherwise constant-depth water body of infinite horizontal extent.

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SOME STEADY STATE EXACT SOLUTIONS OF THE WIND-DRIVEN CIRCULATION EQUATIONS FOR A RIGID-LID SHALLOW LAKE

by Aldo Giorgini[1]

INTRODUCTION

The description of wind-driven circulation in small lakes is usually obtained by numerical integration of the linearized Navier-Stokes equations. In the case of real lakes, one can hardly think that a closed form analytical solution is possible to obtain.

Several researchers [1,2,3,4] have therefore developed computer programs, based on well documented numerical techniques, for the solution of the problem, but little has been reported about the accuracy of such solutions, and still less about the comparisons (and relative computational efficiency) of the several techniques.

The major handicap to the accomplishment of such comparisons is constituted by the fact that no fully three-dimensional closed-form solutions of the equations seem to be known [5].

In the present paper a method is presented for the generation of exact steady state solutions which, albeit not necessarily similar to anything that may be observed in the field, could be observed in the laboratory, and have the advantage of being fully three-dimensional and therefore to

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constitute an excellent standard for comparison with numerical solutions, and with laboratory experiments.

The method, based on an analogous technique successfully applied to the Navier-Stokes equations by Travis-Giorgini [6] relaxes the boundary condition of the shear at the free surface making it become one of the unknowns of the problem, while dictating as a condition a field that is usually part of the solution.

GOVERNING EQUATIONS

The purpose of this section is to present the governing differential equations for the steady-state, wind driven circulation of shallow homogeneous lakes, as formulated by Ligget and Hadjitheodorou [1].

The geometry of the lake is given in Fig. 1. The symbols used are

u, v, w	the x,y, and z-component of the water velocity field;
μ	the (eddy) viscosity of water (assumed constant);
γ	the density of water;
p	the pressure field in the lake;
τ_x, τ_y	the x and y component of the wind generated stress at the surface.

The equations for the dynamics of the lake, under the hypotheses of rigid lid, shallowness and no Coriolis force, are

$$\mu \frac{\partial^2 p}{\partial z^2} = \frac{\partial p}{\partial x} \qquad \dots\dots\dots \qquad (1)$$

$$\mu \frac{\partial^2 v}{\partial z^2} = \frac{\partial p}{\partial y} \qquad \dots\dots\dots \qquad (2)$$

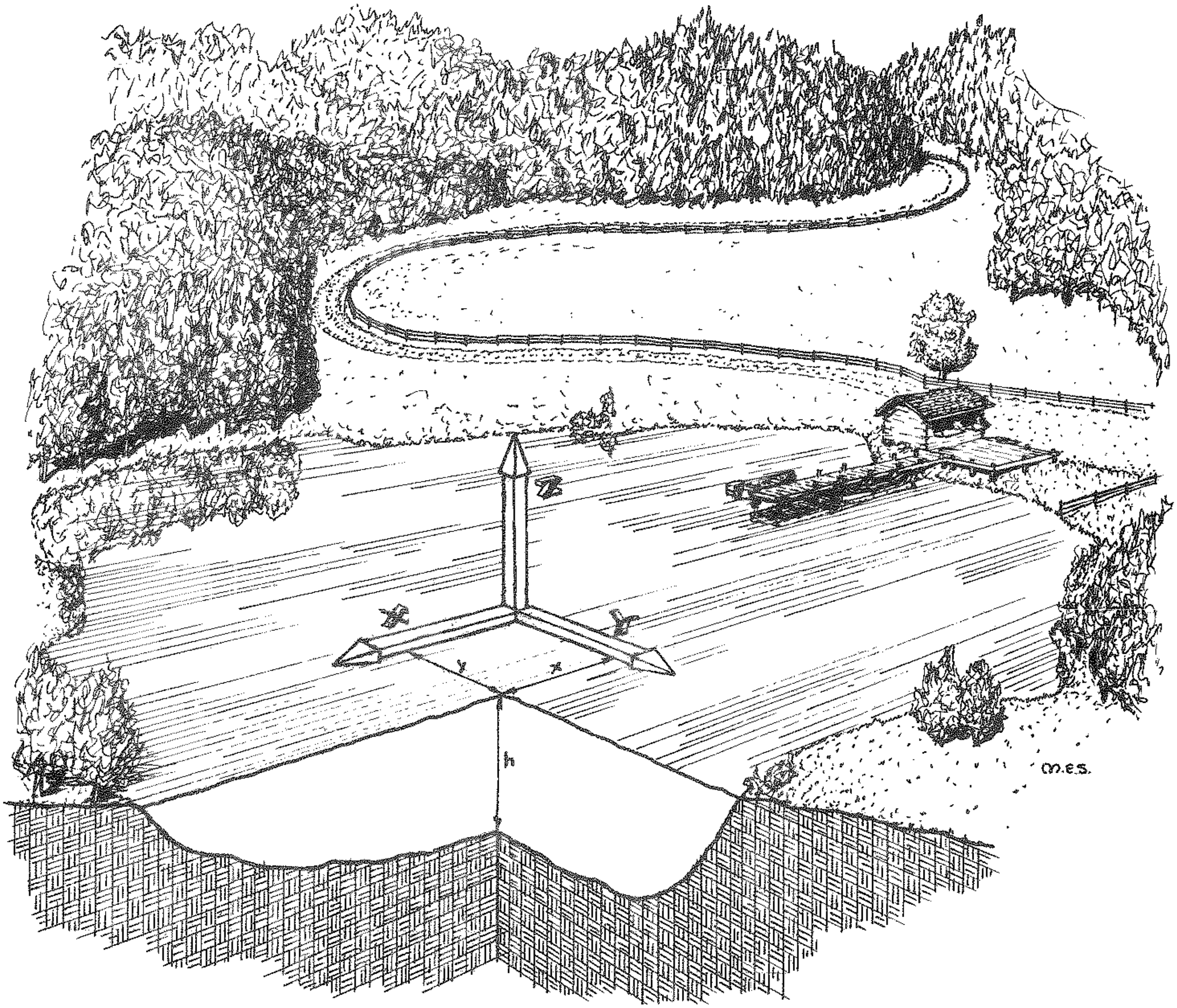


Fig. 1 - Coordinates' definition for the lake geometry.

$$\frac{\partial p}{\partial z} = -\gamma \quad \ldots\ldots (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \ldots\ldots (4)$$

A detailed derivation of these equations is found in [1].

The boundary conditions are

$$u(-h) = v(-h) = w(-h) = w(0) = 0 \quad \ldots\ldots (5)$$

$$\mu \left.\frac{\partial u}{\partial z}\right|_{z=0} = \tau_x \;,\quad \mu \left.\frac{\partial v}{\partial z}\right|_{z=0} = \tau_y \quad \ldots\ldots (6)$$

Equation 3 is solved outright as

$$p = -\gamma z + p_o \quad \ldots\ldots (7)$$

where p_o is the pressure distribution at the free surface. Under the rigid lid approximation, the pressure p_o is equivalent to the pressure distribution caused by the lake setup.

A horizontal length scale R, a vertical length scale H, and a velocity scale V are now defined and Equations 1,2, and 4 recast in terms of the non-dimensional variables so defined:

$$\frac{z}{H} \to z \;,\qquad \frac{x}{R} \to x \;,\qquad \frac{y}{R} \to y \;,\qquad \frac{h}{H} \to h$$

$$\frac{u}{V} \to u \qquad \frac{v}{V} \to v \qquad \frac{w}{V} \to w$$

$$\frac{p_o}{\mu \frac{V}{H}} \to p_o \qquad \frac{\tau_x}{\mu \frac{V}{H}} \to \tau_x \qquad \frac{\tau_y}{\mu \frac{V}{H}} \to \tau_y$$

with the further definition of $\lambda = H/R$.

The nondimensional set of equations becomes

$$\frac{\partial^2 u}{\partial z^2} = \lambda \frac{\partial p_o}{\partial x} \quad \ldots\ldots (8)$$

$$\frac{\partial^2 v}{\partial z^2} = \lambda \frac{\partial p_o}{\partial y} \quad \ldots\ldots\ldots (9)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\lambda}\frac{\partial w}{\partial z} = 0 \quad \ldots\ldots\ldots (10)$$

with boundary conditions

$$u(-h) = v(-h) = w(-h) = w(0) = 0 \quad \ldots\ldots\ldots (11)$$

$$\left.\frac{\partial u}{\partial z}\right|_o = \tau_x \qquad \left.\frac{\partial v}{\partial z}\right|_o = \tau_y \quad \ldots\ldots\ldots (12)$$

A first integration of Equation 8 from 0 to z yields

$$\frac{\partial u}{\partial z} = \lambda \frac{\partial p_o}{\partial x} z + \tau_x \quad \ldots\ldots\ldots (13)$$

and a further integration from -h to z yields

$$m = \frac{\lambda}{2} \frac{\partial p_o}{\partial x} (z^2 - h^2) + \tau_x (z + h) \quad \ldots\ldots\ldots (14)$$

Analogously, from Equation 9 the expression for v could be derived

$$v = \frac{\lambda}{2} \frac{\partial p_o}{\partial y} (z^2 - h^2) + \tau_y (z + h) \quad \ldots\ldots\ldots (15)$$

The velocity field if therefore known (once τ_x and τ_y are given) unless for the surface pressure field p_o. The equation for the field p_o is obtained by applying the boundary conditions of w in Equation 10, that is, by integrating Equation 10 from -h to 0. The result is

$$\frac{\partial}{\partial x}\int_{-h}^{0} u dz + \frac{\partial}{\partial y}\int_{-h}^{0} v\, dz = 0 \quad \ldots\ldots\ldots (16)$$

which is identically satisfied if

$$\left.\begin{aligned} \int_{-h}^{0} dz &= \frac{\partial \psi}{\partial y} \\ \int_{-h}^{0} dz &= -\frac{\partial \psi}{\partial x} \end{aligned}\right\} \quad \ldots\ldots\ldots (17)$$

Upon performance of the integrals suggested in Equations 17 and Equations 14 and 15 the following is obtained

$$\frac{1}{2}\,\tau_x\,h^2 - \frac{1}{3}\,\lambda\,\frac{\partial p_o}{\partial x}\,h^3 = \frac{\partial \psi}{\partial y} \quad \ldots\ldots\ldots\ldots (18)$$

$$\frac{1}{2}\,\tau_y\,h^2 - \frac{1}{3}\,\lambda\,\frac{\partial p_o}{\partial y}\,h^3 = -\,\frac{\partial \psi}{\partial x} \quad \ldots\ldots\ldots\ldots (19)$$

which is a system of simultaneous partial differential equations in the two x-and-y-dependent fields p_o and ψ [the streamfunction of the horizontal specific flowrate field].

The method (or trick*) here suggested is the following. Instead of solving Equations 18 and 19 for p_o and ψ, given the functions h, τ_x, and τ_y, solve the same equations for τ_x and τ_y after having specified suitable functions for p_o and ψ.

It is obvious that this process hardly requires any labor in terms of "solving" Equations 18 and 19, since τ_x and τ_y are already explicitly expressed in terms of p_o and ψ. This part will be illustrated first by means of one example.

EXAMPLE: LAKE WITH GAUSSIAN BATHYMETRY

Assume that the bathymetry of the lake is given by

$$h = e^{-(x^2+y^2)} \quad \ldots\ldots\ldots\ldots (20)$$

and give the plausible functions

$$\psi = x\,h^2 \quad \ldots\ldots\ldots\ldots (21)$$

$$p_o = \alpha\,\frac{y}{h} \quad \ldots\ldots\ldots\ldots (22)$$

where α is an arbitrary constant.

The Author remembers a comment made by G. E. Uhlenbeck in a lecture at Colorado University in the Summer 1965: "The difference between a trick and a method lies only in the frequency of its applicability."

By substituting Equations 21 and 22 into Equations 18 and 19 we obtain

$$\tau_x = 4xy \left(\frac{\alpha\lambda}{3} - 2\right) \qquad (23)$$

$$\tau_y = \frac{2}{3}\,\alpha\,\lambda\,(1 + 2y^2) - 2(1 - 4x^2) \qquad (24)$$

Since α is an arbitrary constant, we can set it so as to have $\tau_x = 0$, that is

$$\alpha = \frac{6}{\lambda} \qquad (25)$$

to get

$$\tau_y = 2 + 8\,(x^2 + y^2) \qquad (26)$$

Therefore the solution of the shallow lake equations with the above shear conditions and with Gaussian bathymetry is

$$u = 6\,x\,y\,h\,\left(\left(\frac{z}{h}\right)^2 - 1\right) \qquad (27)$$

$$v = 3h\,(1 + 2y^2)\left(\left(\frac{z}{h}\right)^2 - 1\right) + 2h\,(1 + 4(x^2 + y^2)\left(\frac{z}{h} + 1\right) \qquad (28)$$

$$p_o = \frac{6}{\lambda}\,\frac{y}{h} \qquad (29)$$

At the free surface the velocity field is

$$u_o = -6\,x\,y\,h \qquad (30)$$

$$v_o = h(8x^2 + 2y^2 - 1) \qquad (31)$$

The streamlines of the surface velocity field satisfy the equation

$$y^2 = c\,x^{-2/3} - x^2 + 1/2 \qquad (32)$$

where c is an arbitrary constant. For $c = 0$ the streamline becomes a circled of radius $\sqrt{2}/2$. Furthermore the locus of the points with $u = 0$ is constituted by the two axes and the locus of the points with $v = 0$ is an ellipse with semiaxes $\sqrt{2}/4$ and $\sqrt{2}/2$ respectively.

The stream function of the horizontal specific flowrate is given by Eq. 21 (which shows immediately that x=0 is a streamline for the vertically averaged flow), and the vertically averaged velocity field is therefore

$$\overline{u} = \frac{1}{h}\frac{\partial\psi}{\partial y} = -4xhy \quad \dots\dots\dots\dots \quad (33)$$

$$\overline{v} = -\frac{1}{h}\frac{\partial\psi}{\partial x} = (4x^2-1)h \quad \dots\dots\dots\dots \quad (34)$$

which show that $\overline{u}$ is zero on the two axes and that $\overline{\ }$ is zero on the straight lines $x = \pm 1/2$.

Figure 2 presents some views of the velocity field in the Gaussian bathymetry lake.

THE TRICK IN THE EXAMPLE

The expressions (21) and (22) give the impression of a gratuitous accident. Actually some rationale justifies them.

We assume first that we are interested in the one-directional wind case only, that is the τ_x=0 case. Assuming then that we are dealing with a bathymetry h symmetric with respect to x and y, it is reasonable to look for solutions ψ and p_o of the form

$$\psi = f(x)F(h) \quad \dots\dots\dots\dots \quad (35)$$

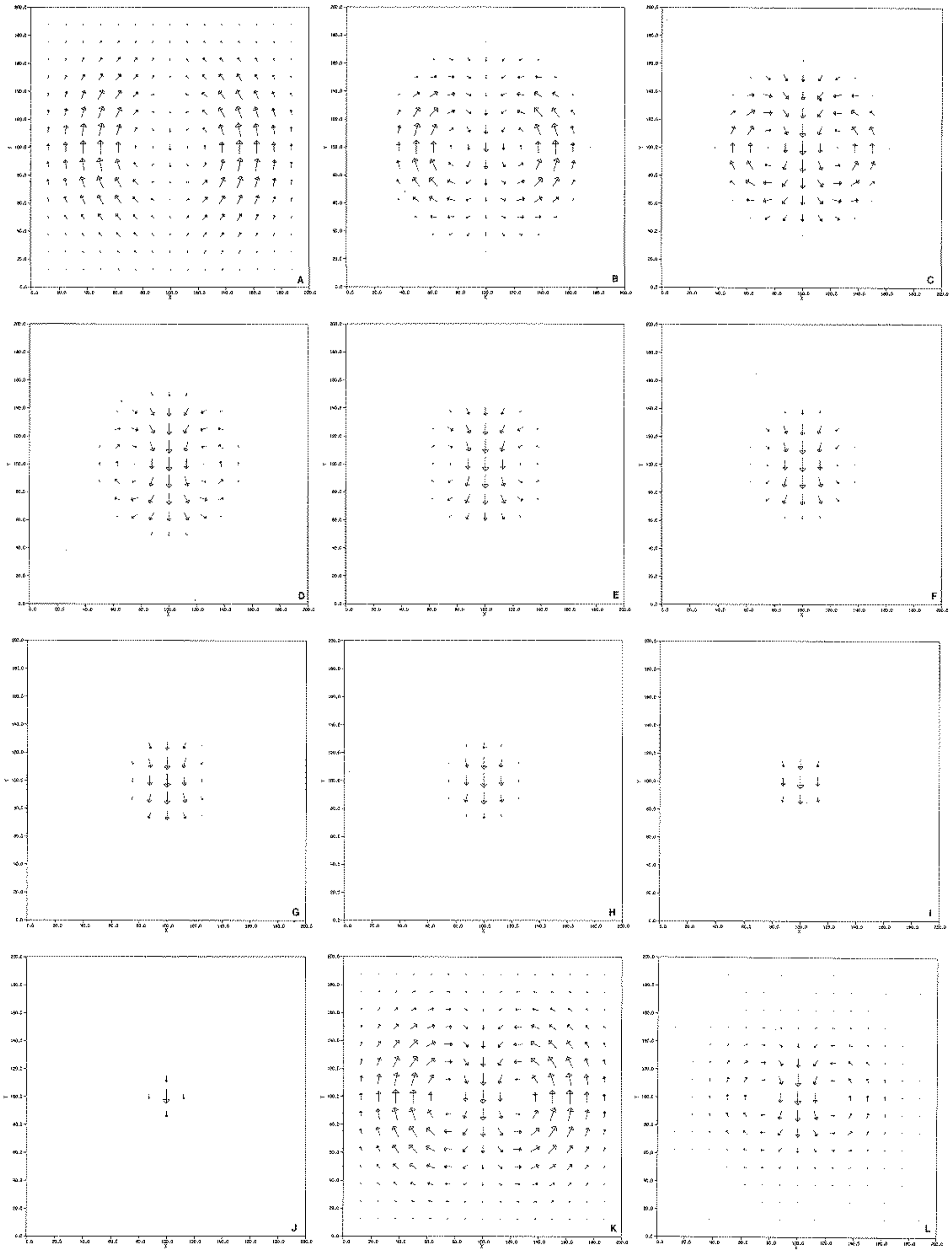
$$p_o = g(y)G(h) \quad \dots\dots\dots\dots \quad (36)$$

where f and g can be anticipated to be odd functions of their respective arguments.

Substitution of (35) and (36) into (18) and (19) yields

$$-\frac{1}{3}\lambda h^3 gG' \frac{\partial h}{\partial x} = fF' \frac{\partial h}{\partial y} \quad (37)$$

$$\frac{1}{2}\tau_y h^2 - \frac{1}{3}\lambda h^3[g'G + gG' \frac{\partial h}{\partial y}] = -f'F - fF' \frac{\partial h}{\partial x} \quad (38)$$



2 - Velocity field in the Gaussian bathymetry lake with variable surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.

Assuming now that h is a function of $r = \sqrt{x^2+y^2}$ only, the above expressions can be written as

$$-\frac{1}{3}\lambda h^3 gG'x = fF'y \quad \ldots\ldots (39)$$

$$\frac{1}{2}\tau_y h^2 - \frac{1}{3}\lambda h^3\left[g'G + gG'\frac{y}{r}\frac{\partial h}{\partial r}\right] = -f'F - fF'\frac{x}{r}\frac{\partial h}{\partial r} \quad \ldots\ldots (40)$$

Furthermore, (39) can be rewritten as

$$-\frac{1}{3}\lambda h^3\frac{G'}{F'} = \frac{fy}{gx}$$

which, according to our stipulations about the functions f, g, F, and G, can only hold if f=ax, g=by, $-\frac{1}{3}\lambda h^3\frac{G'}{F'} = \frac{a}{b}$. Without loss of generality we could put a=b=1 and obtain

$$G' = -\frac{3}{\lambda}\frac{F'}{h^3} \quad \ldots\ldots (41)$$

These results modify (40) into

$$\frac{1}{2}\tau_y h^2 - \frac{1}{3}\lambda h^3 G + r\frac{\partial h}{\partial r}F' = -F \quad \ldots\ldots (42)$$

We can now eliminate G from (41) and (42) by taking the derivative of (42) and substituting into it the values of G' and of G obtained from (41) and (42), respectively. By so doing we obtain

$$\left[r\frac{\partial h}{\partial r}\right]F'' + \left[2 + h^3\frac{\partial}{\partial h}\left(\frac{1}{h^3}\left[r\frac{\partial h}{\partial r}\right]\right)\right]F' - 3\frac{F}{h} + \frac{h^3}{2}\frac{\partial}{\partial h}\left(\frac{\tau_y}{h}\right) = 0 \quad \ldots\ldots (43)$$

In this expression the group $\left[r\frac{\partial h}{\partial r}\right]$ should be written in terms of h, when the bathymetry is known.

(43) is an equation in h, F, and τ_y. It is therefore sufficient to specify any two of these functions to obtain the third.

In order to make the following discussion a little more colloquial and a little less symbolic, let us recall the meanings of the three variables.

The variable h (a function of r) stands for the bathymetry of the lake. The variable τ_y stands for the one-directional wind action on the surface of the lake, that is for the forcing action which causes the lake circulation. The variable F stands for the function that multiplied by x yields the stream function of the horizontal specific flowrates, that is, for the flow or circulation within the lake.

The most interesting case is the one of finding F, given h and τ_y. This is the most difficult too, and we will start from it.

A. FIND the Circulation (F) - caused by a given forcing (τ_y) on a lake of given bathymetry (h)

The most interesting subcase is the one in which τ_y is a constant. We will consider only this subcase. Equation (43) becomes

$$\left[r \frac{\partial h}{\partial r}\right] F'' + \left[2 + h^3 \frac{\partial}{\partial h}\left\{\frac{1}{h^3}\left[r \frac{\partial h}{\partial r}\right]\right\}\right] F' - 3\frac{F}{h} = \frac{\tau_y}{2} h \quad \ldots\ldots\ldots\ldots (44)$$

A general solution of (44) with unspecified bathymetry seems impossible. Therefore we will look for special bathymetric classes.

(a) the bathymetry is of the form

$$h = \frac{1}{1+r^n} \qquad \therefore \; r\frac{\partial h}{\partial r} = -nh(1-h)$$

In this case (44) becomes

$$-nh^2(1-h)F'' + h(2+2n-nh)F' - 3F = \frac{\tau_y}{2} h^2 \quad \ldots\ldots\ldots\ldots (45)$$

A particular solution of (45) is

$$F = \frac{\tau_y}{2(2n+1)} h^2 \quad \ldots\ldots\ldots\ldots (46)$$

By substituting (46) into (41) we can find

$$G = \frac{3\tau_y}{\lambda(2n+1)}\left(\frac{1}{h} + n\right) \quad \ldots\ldots\ldots\ldots (47)$$

The solution of this case can be written down in the following table.

TABLE I

Bathymetry $h = \dfrac{1}{1+r^n}$

where $r = \sqrt{x^2+y^2}$

Shear τ_y = constant

(A) $\psi = \dfrac{\tau_y x}{2(2n+1)} h^2$

(B) $p_o = \dfrac{3\tau_y y}{\lambda(2n+1)} \left(\dfrac{1}{h} + n\right)$

(C) $u = \dfrac{3n\tau_y}{2(2n+1)} xyr^{n-2}(z^2-h^2)$

(D) $v = \dfrac{3\tau_y}{2(2n+1)} \left[\dfrac{1}{h} + n + ny^2r^{n-2}\right](z^2-h^2) + \tau_y(z+h)$

(E) $\overline{u} = \dfrac{\partial\psi}{\partial x} = -n\tau_y \dfrac{xyh(1-h)}{r^2(2n+1)}$

(F) $\overline{v} = -\dfrac{\partial\psi}{\partial y} = -\dfrac{\tau_y h}{2n+1}\left(\dfrac{1}{2} - n\dfrac{x^2(1-h)}{r^2}\right)$

(G) $w = -\lambda\tau_y \dfrac{n(n+2)}{2(2n+1)} \dfrac{y}{r^2} \dfrac{1-h}{h} z(z+h)^2$

Each equation of Table I has been derived in the following fashion:

(A) : by substituting (46) into (35);

(B) : by substituting (47) into (36);

(C) : by substituting (B) into (14);

(D) : by substituting (B) into (15);

(E) : by substituting (A) into definition of $\overline{u}$;

(F) : by substituting (A) into definition of $\overline{v}$;

(G) : by substituting (B) into (A-7) of APPENDIX A according to the outline of APPENDIX B.

(b) Exponential bathymetry

$$h = e^{-r^n} \qquad r = \frac{\partial h}{\partial r} = nh\,\ln h$$

This case has as particular case the Gaussian bathymetry.

In this case (44) becomes

$$nh^2 \ln h\ F'' + h(2 + n[1-2\ln h])F' - 3F = \frac{\tau_y}{2} h^2$$

An analytic solution of this case is not obvious. Nevertheless, if an exponential bathymetry were essential for some special problem, (48) could be easily solved numerically.

B. FIND the Forcing (τ_y) - that causes a given circulation (F) into a lake of given bathymetry (h)

Recalling the form of F in A we will assume that $F = \alpha h^\beta$, from which, by substituting into (41), we can get

$$G = -\frac{3}{\lambda}\alpha\beta\frac{h^{\beta-3}}{\beta-3} + \gamma$$

where γ is any constant. Substitution of these into (42) yields

$$\tau_y = -2\alpha h^{\beta-2}\left[\frac{2\beta-3}{\beta-3} + \beta\frac{r}{h}\frac{\partial h}{\partial r}\right] + \frac{2}{3}\lambda\gamma h \quad (\text{for } \beta\neq 3) \qquad \text{.........(49a)}$$

In particular, when $\beta=2$, $\gamma = 6\alpha n/\lambda$ for the case $h = 1/(1+r^n)$ τ_y becomes a constant.

The case $\beta=3$ needs a special derivation. The result is

$$\tau_y = -6\alpha h\left(\ln h + \frac{r}{h}\frac{\partial h}{\partial r}\right) + \left(\frac{2}{3}\lambda\gamma - 2\alpha\right)h \qquad (\text{for } \beta=3) \qquad \text{.........(49b)}$$

The example of the lake with Gaussian bathymetry can be seen to be a particular case of (49a) when $\beta=2$ and $h = e^{-r^2}$.

The next case, the one of finding the bathymetry that yields a given circulation under a given forcing action, will not be considered in the present context.

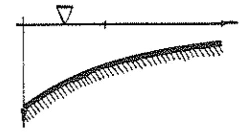
COMMENTS ON THE EXACT SOLUTIONS FOR THE LAKE CIRCULATION

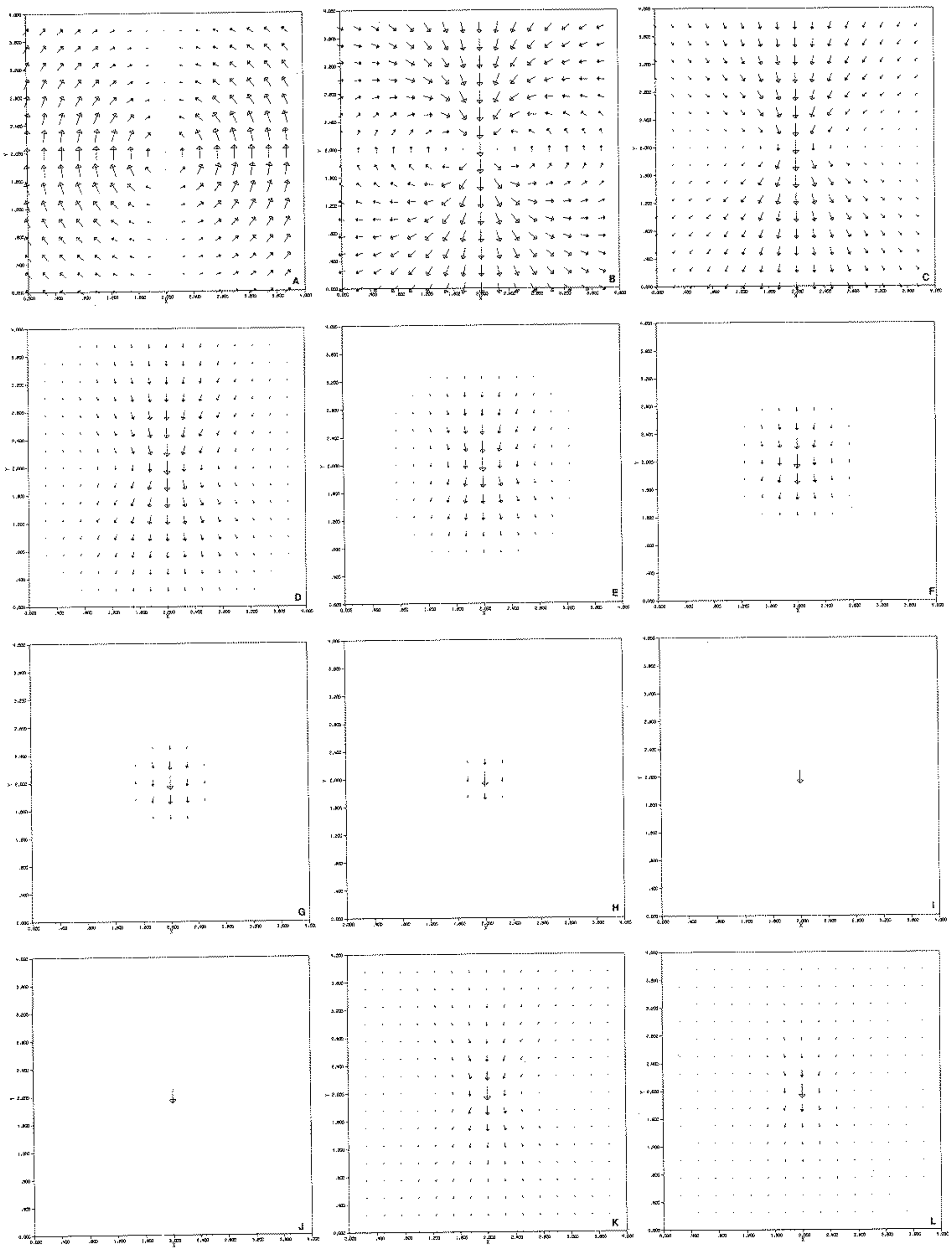
The results of Table I have been plotted for several bathymetries in the figures from 3 to 10. Figure 3 corresponds to a bathymetry with a cusp at the origin (n=1). The following figures correspond to belly-shaped bathymetries which tend to become flatter as the figure number increases. For the last figure of this series, Figure 10, corresponding to n=32, the bathymetry is almost like a cylindrical cavity.

It is to be anticipated that for large values of n the exact solutions of the lake circulation equations (which are approximations of the Navier-Stokes equations) may not be physically acceptable. In fact, one of the assumptions (small slope of the bathymetry, or quasi-planar flow) is no longer satisfied by the geometry. That the case is so, is apparent from Figure 10 (n=32) where the flow in the "cylindrical" lake seems to be one-directional. Physically we may expect that this be not the case. In fact the vertical part of the cylindrical cavity, not contemplated by the approximate model, would have a definite curving effect on the flow field.

Another observation is necessary: for n=1 and n=2 the volume of the lake is infinite, regardless of how shallow it be far from its center. These two cases have nevertheless been presented in order to allow the reader to have a complete perspective of the range of the solutions.

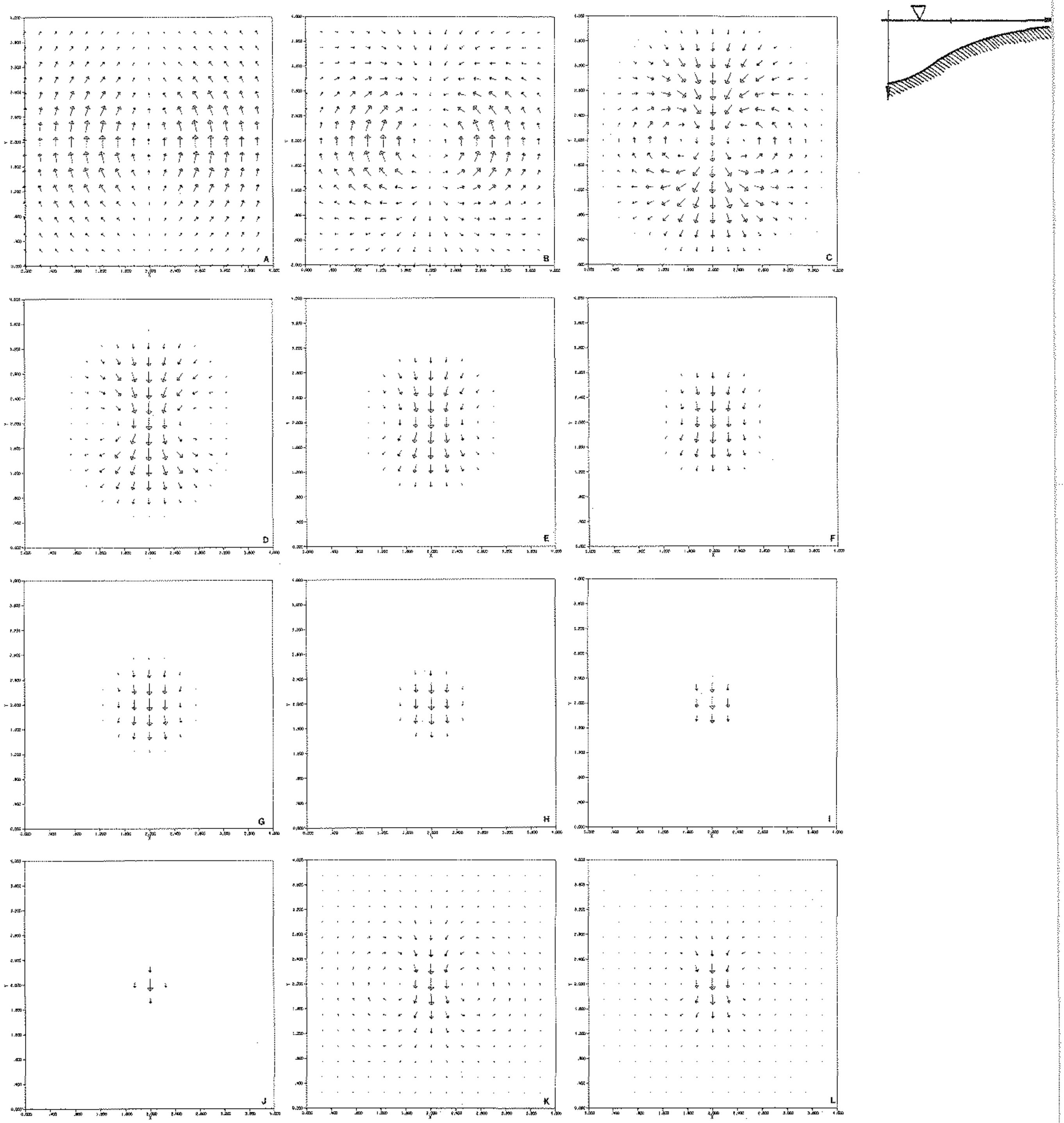
Figure 11 presents the pressure distribution over the surface of the lake for the case of Fig. 6, that is for the case $h = 1/(1+r^4)$. The isobaric lines drawn in the figure could be considered as the contour lines for the lake setup caused by the wind. It is interesting to point out that the slope of the setup tends to become steeper as the water becomes shallower. This is another limit for the validity of the exact solution.





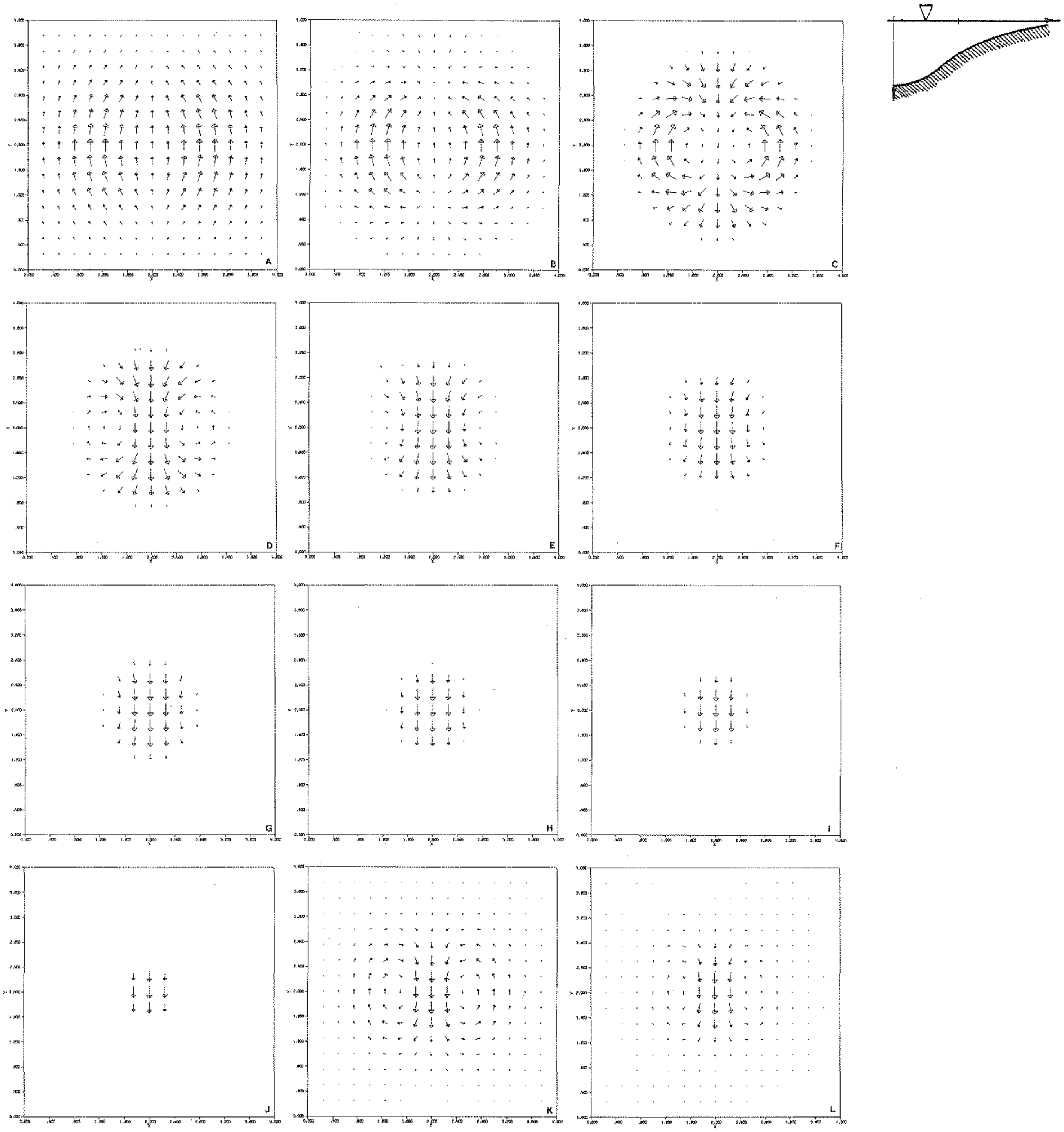
LAKE BATHYMETRY: $h = \frac{1}{1 + r}$

Fig. 3 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.



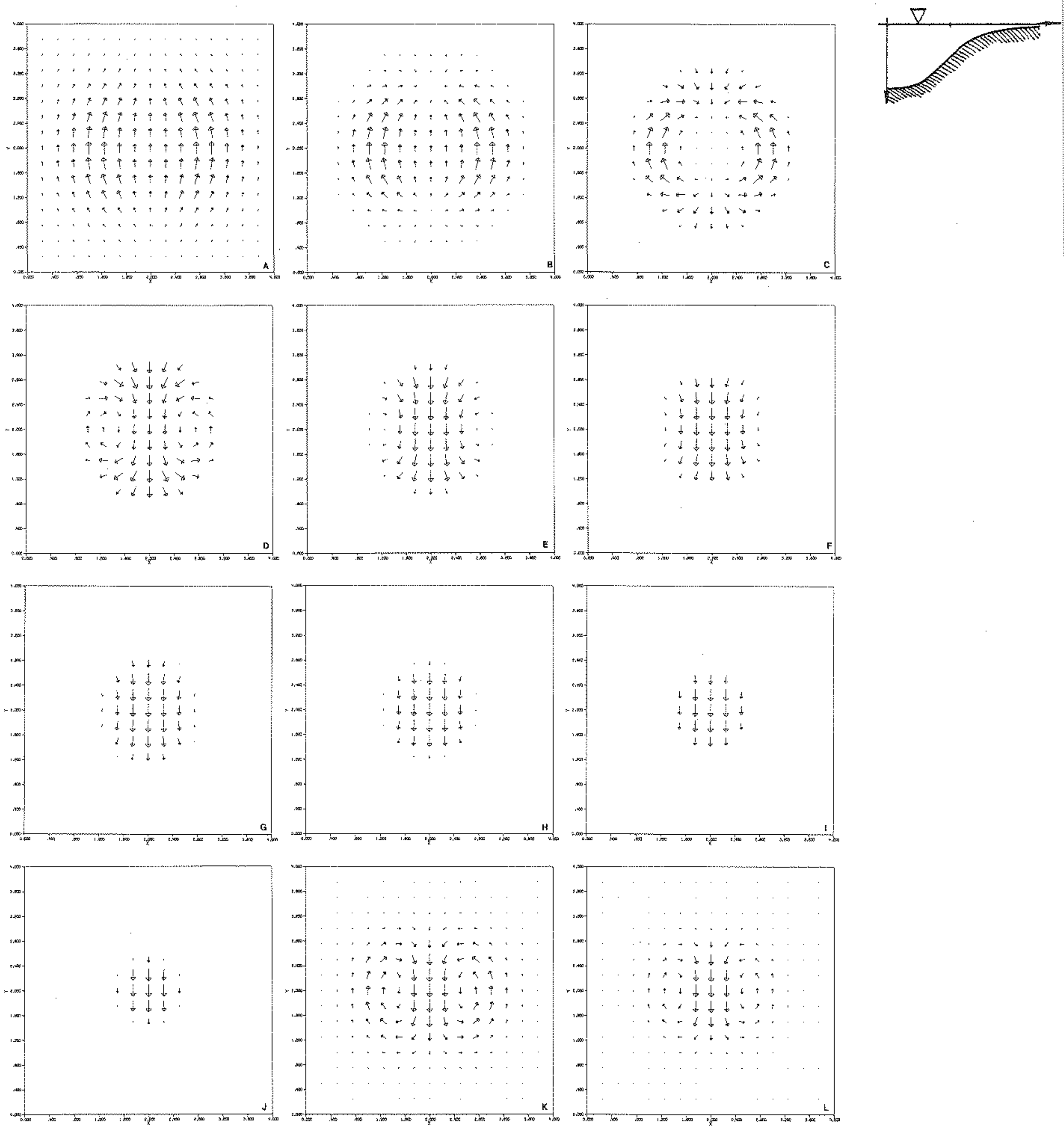
$$\text{LAKE BATHYMETRY:} \quad h = \frac{1}{1 + r^2}$$

Fig. 4 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.



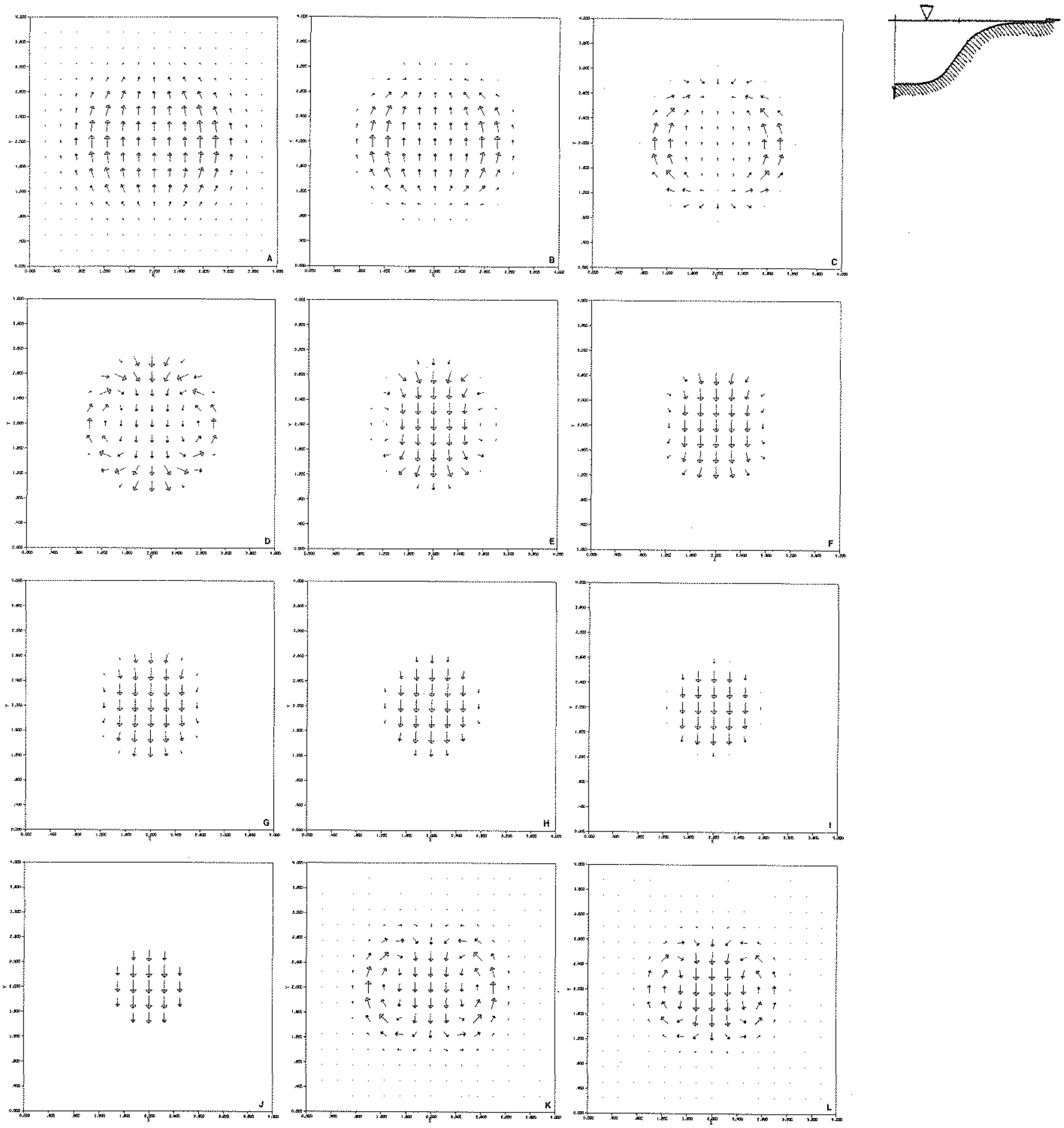
$$\text{LAKE BATHYMETRY:} \quad h = \frac{1}{1 + r^3}$$

Fig. 5 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.



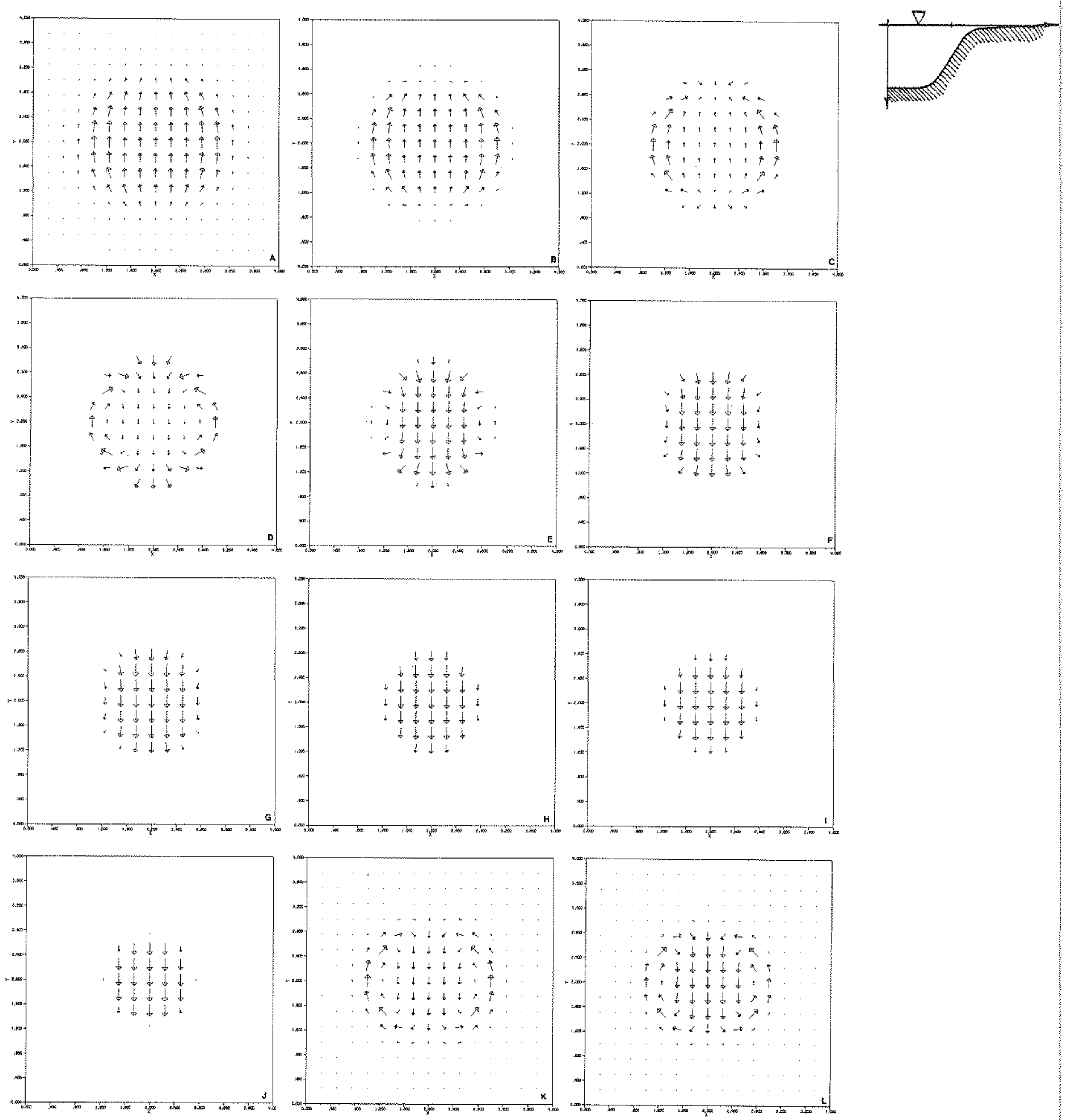
LAKE BATHYMETRY: $h = \dfrac{1}{1 + r^4}$

Fig. 6 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.



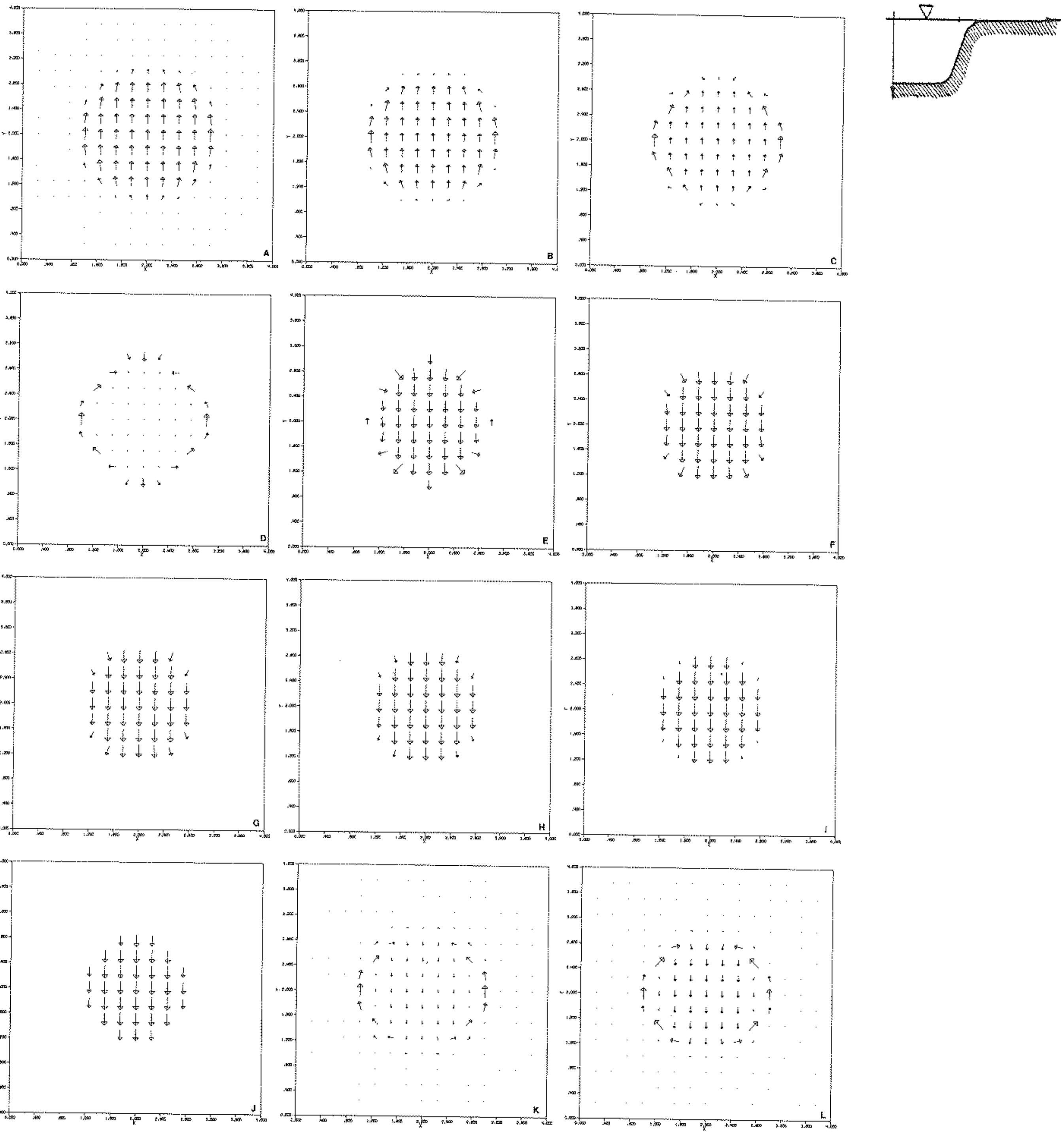
LAKE BATHYMETRY: $h = \dfrac{1}{1 + r^6}$

Fig. 7 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.



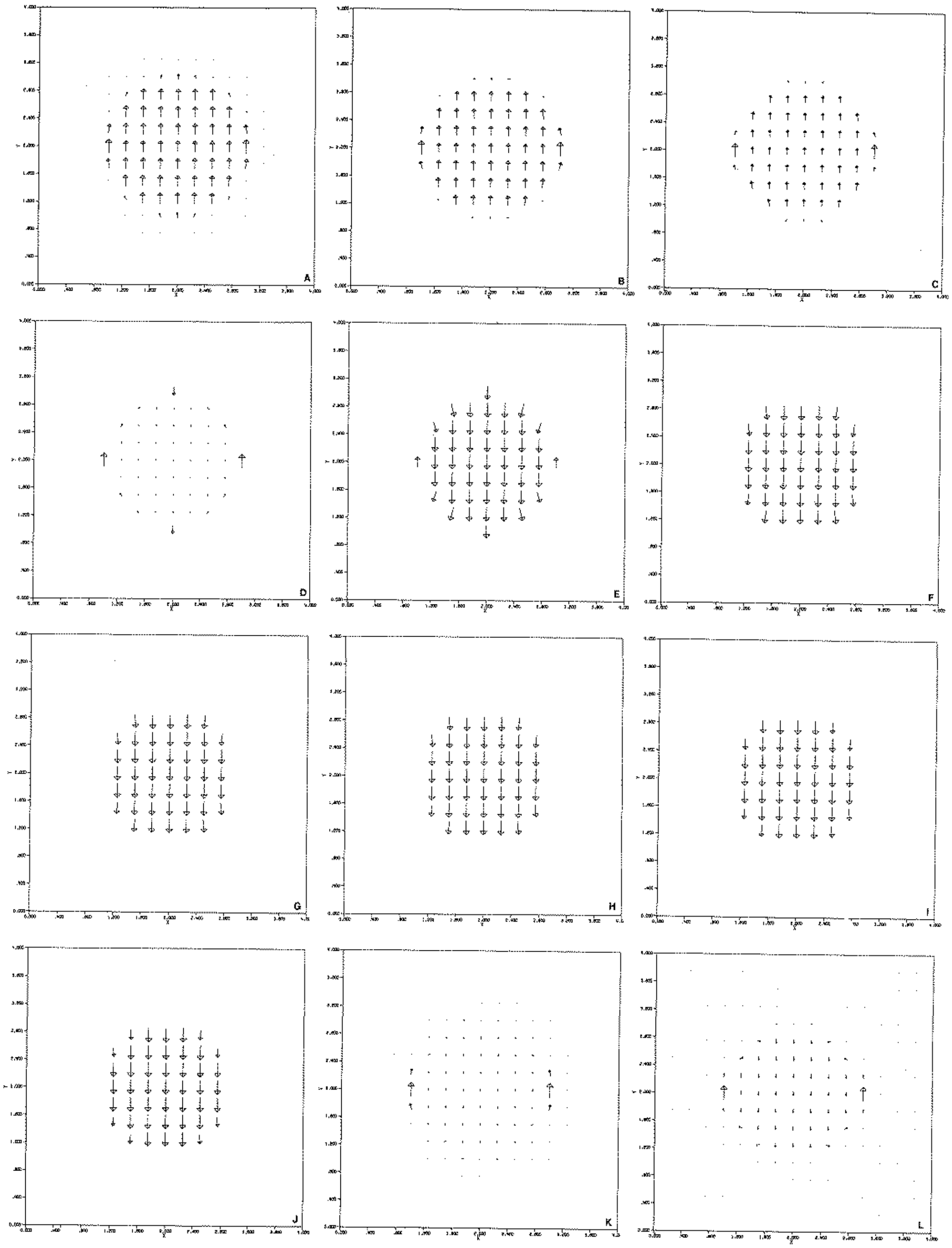
LAKE BATHYMETRY: $h = \dfrac{1}{1 + r^8}$

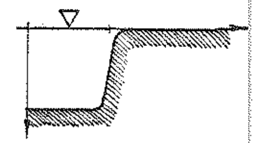
Fig. 8 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.



LAKE BATHYMETRY: $h = \dfrac{1}{1 + r^{16}}$

Fig. 9 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.





LAKE BATHYMETRY: $h = \dfrac{1}{1 + r^{32}}$

Fig. 10 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.

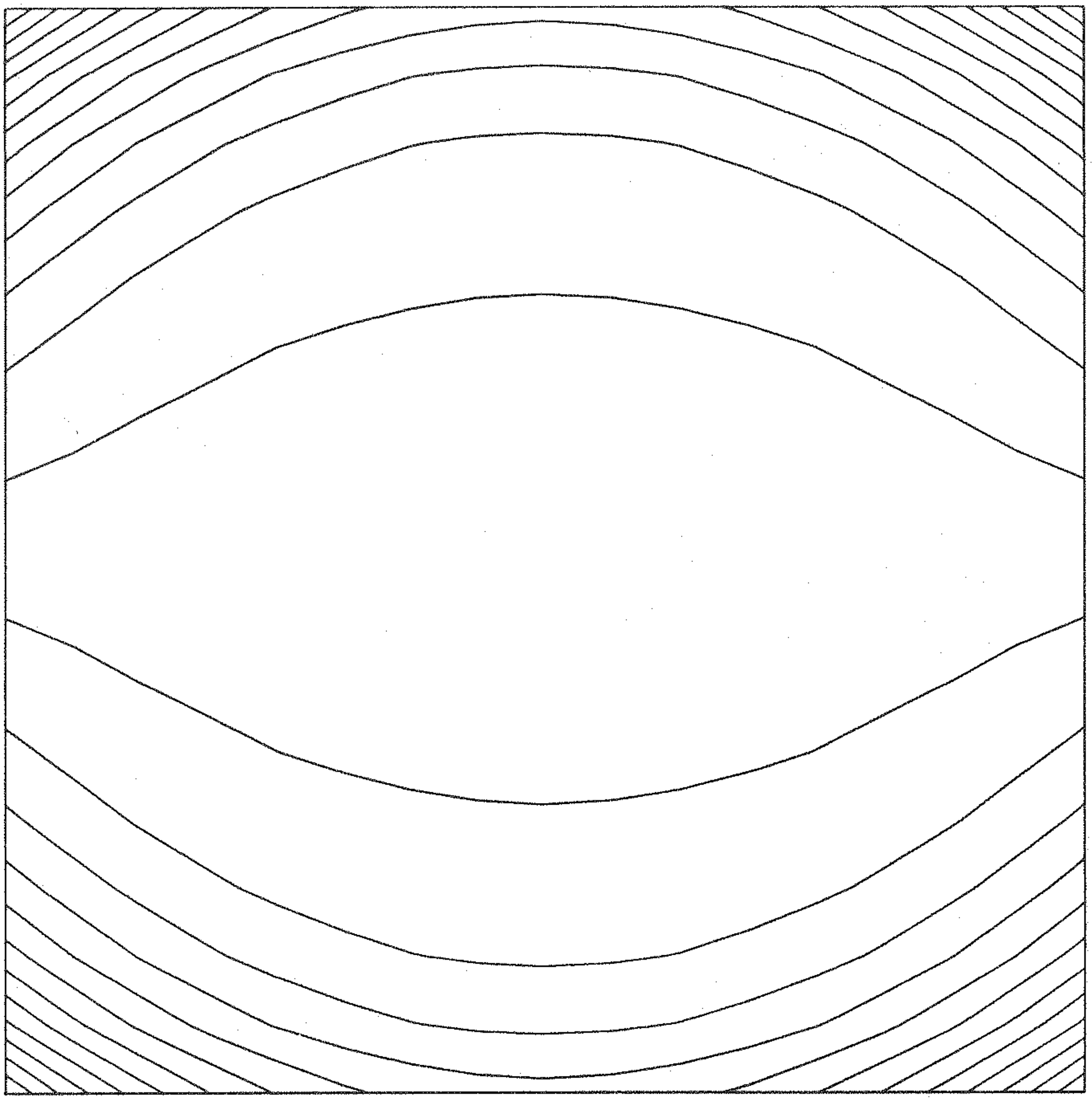


Fig. 6 - Pressure distribution abroad the lake for the bathymetry of Fig. 7. The lines are isobaric lines drawn for equal increment of pressure. They may be considered the contour lines for the lake setup.

In fact, as soon as the setup is allowed to grow beyond a small fraction of the depth, the rigid-lid approximation is no longer realistic.

GENERALIZATION OF THE TRICK

The approach presented above can be somehow generalized when a particular problem is under investigation. The concern of the previous problem was the circulation in a lake, that is a water pocket. What we want to concern ourselves with now is the problem of the circulation in a pool or around a hill in an otherwise constant bathymetry. For this case we assume therefore a known bathymetry

$$h = 1 + \frac{\varepsilon}{1+r^n} \qquad \text{for } |\varepsilon| < 1 \qquad \text{(50)}$$

which yields a pool for $\varepsilon>0$ and a hill for $\varepsilon<0$.

We follow the lines of the previous example and, since we had previously found a function $F = \alpha h^\beta$ for a bathymetry $1/(1+r^n)$, we tentatively assume a function

$$F = Ah^2 + Bh + C \qquad \text{(51)}$$

for this more general case.

By inserting (51) into (41) and integrating once we obtain

$$G = \frac{3}{\lambda}\left(\frac{2A}{h} + \frac{B}{2h^2} + D\right) \qquad \text{(52)}$$

By inserting (51) and (52) into (42) we obtain

$$Dh^3 + (A - \frac{\tau_y}{2})h^2 - \frac{B}{2}h - C = (2Ah+B)r\frac{dh}{dr} \qquad \text{(53)}$$

With the given bathymetry the expression $r\frac{dh}{dr}$ becomes

$$r\frac{\partial h}{\partial r} = \frac{h}{\varepsilon}(1-h)(1+\varepsilon-h) \qquad \text{(54)}$$

Substitution of (54) into (53) yields

$$D = 2A\,\frac{n}{\varepsilon}$$

$$A - \frac{\tau_y}{2} = B\,\frac{n}{\varepsilon} - 2A\,\frac{n}{\varepsilon}\,(2+\varepsilon)$$

$$-\frac{B}{2} = -B\,\frac{n}{\varepsilon}\,(2+\varepsilon) + 2A\,\frac{n}{\varepsilon}\,(1+\varepsilon)$$

$$-C = B\,\frac{n}{\varepsilon}\,(1+\varepsilon)$$

The solution of this system of equations in A, B, C, D is

$$A = \frac{\tau_y \varepsilon}{2}\,\frac{4n + \varepsilon(2n-1)}{4n^2(3+3\varepsilon+\varepsilon^2) - \varepsilon^2} \quad \text{. . . . . (55)}$$

$$B = \frac{4n(1+\varepsilon)}{4n+2n\varepsilon-\varepsilon}\;A = \frac{\tau_y \varepsilon}{2}\,\frac{4n(1+\varepsilon)}{4n^2(3+3\varepsilon+\varepsilon) - \varepsilon^2} \quad \text{. . . . . (56)}$$

$$C = -n\,\frac{1+\varepsilon}{\varepsilon}\,B \quad \text{. . . . . (57)}$$

$$D = 2\,\frac{n}{\varepsilon}\,A = n\tau_y\,\frac{4n + \varepsilon(2n-1)}{4n^2(3+3\varepsilon+\varepsilon^2) - \varepsilon^2} \quad \text{. . . . . (58)}$$

With the knowledge of these constants we can write ψ and p_o as

$$\psi = x(Ah^2 + Bh + C) \quad \text{. . . . . (59)}$$

$$p_o = \frac{3y}{\lambda}\left(\frac{2A}{h} + \frac{B}{2h^2} + D\right) \quad \text{. . . . . (60)}$$

The fields of interest are

$$u = \frac{\lambda}{2}\,\frac{\partial p_o}{\partial x}\,(z^2-h^2) \quad \text{. . . . . (61)}$$

$$v = \frac{\lambda}{2}\,\frac{\partial p_o}{\partial y}\,(z^2-h^2) + \tau_y(2+h) \quad \text{. (62)}$$

where

$$\frac{\partial p_o}{\partial x} = \frac{3nxy}{\varepsilon\lambda}\,\frac{r^{n-2}}{h^2}\,(2A + \frac{B}{h})(h-1)^2 \quad \text{. . . . . (63)}$$

$$\frac{\partial p_o}{\partial y} = \frac{3ny^2}{\varepsilon\lambda}\,(2A + \frac{B}{h})\,\frac{(h-1)^2}{h^2}\,r^{n-2} + \frac{3}{\lambda}\left(\frac{2A}{h} + \frac{B}{2h^2} + D\right) \quad \text{. . . . . (64)}$$

or

$$\frac{\partial p_o}{\partial x} = - \frac{3nxy}{\lambda \varepsilon r^2 h^2} (2A + \frac{B}{h})(1-h)(1+\varepsilon-h) \quad \ldots\ldots\ldots\ldots\ldots\ldots (63a)$$

$$\frac{\partial p_o}{\partial y} = - \frac{3ny^2}{\lambda \varepsilon r^2 h^2} (2A + \frac{B}{h})(1-h)(1+\varepsilon-h) + \frac{3}{\lambda}\left(\frac{2A}{h} + \frac{B}{2h^2} + D\right) \quad \ldots\ldots (64a)$$

or

$$\frac{\partial p_o}{\partial x} = - \frac{3}{\lambda} \frac{xy}{r^2} \left[- \frac{C}{h^3} - \frac{B/2}{h^2} + \frac{A-\tau_y/2}{h} + D\right] \quad \ldots\ldots\ldots\ldots\ldots (63b)$$

$$\frac{\partial p_o}{\partial y} = - \frac{3}{\lambda} \frac{y^2}{r^2} \left[- \frac{C}{h^3} - \frac{B/2}{h^2} + \frac{A-\tau_y/2}{h} + D\right] + \frac{3}{\lambda} \left(\frac{2A}{h} + \frac{B}{2h^2} + D\right) \quad \ldots\ldots (64b)$$

The specific flow rates are

$$q_x = \frac{\partial \psi}{\partial y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (65)$$

$$q_y = - \frac{\partial \psi}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (66)$$

where

$$\frac{\partial \psi}{\partial x} = (Ah^2 + Bh + C) - \frac{n}{\varepsilon} x^2 r^{n-2} (h-1)^2 (2Ah+B) \quad \ldots\ldots\ldots\ldots (67)$$

$$\frac{\partial \psi}{\partial y} = - \frac{n}{\varepsilon} xyr^{n-2} (h-1)^2 (2Ah+B) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (68)$$

and the vertically averaged velocities

$$\bar{u} = q_x/h \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (69)$$

$$\bar{v} = q_y/h \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (70)$$

The calculation of the vertical component of the velocity field can be found in Appendix C.

COMMENTS ON THE EXACT SOLUTIONS FOR THE WIND-DRIVEN CIRCULATION AROUND A SUBAQUEOUS HILL

The cases presented in the figures from Fig. 12 to Fig. 25 contemplate different hill heights (.50 [the top of the hill is half-way between the bottom and the free surface], .99 [the top of the hill is .01 below the free surface]) and different hill shapes (n = 1,2,4,8,16,32).

As we had commented about the lake circulation, we should consider the cases $n \geq 8$ as nonphysical (albeit exact solutions of the approximate equations).

The variety of illustrations is presented so as to give the reader a grasp at the influence of the parameters on the solution.

Worth of notice is the small bottom back flow for the cases $n > 2$.

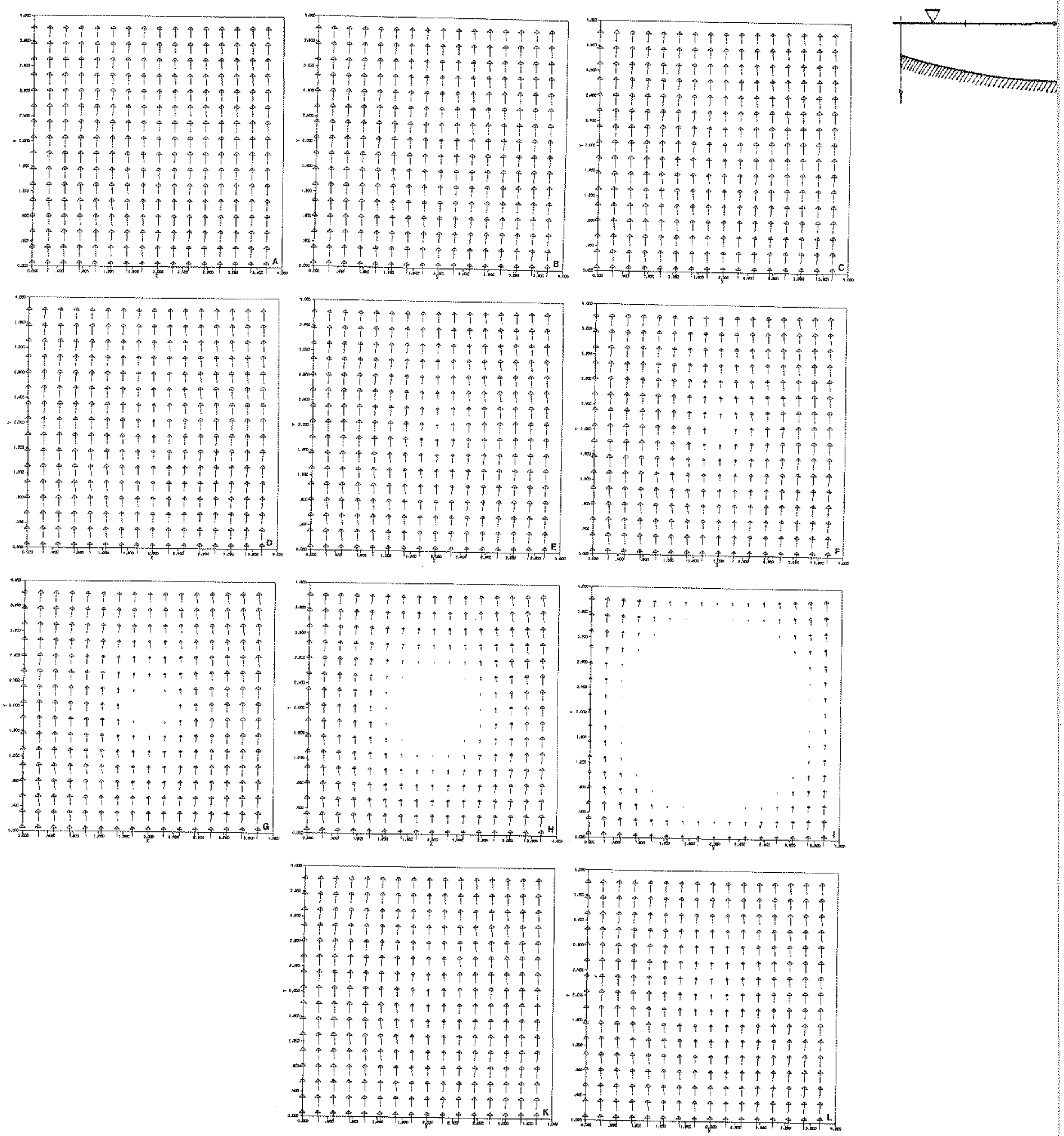
Figure 26 gives an example of pressure distribution across the surface for a subaqueous hill that rises to .875 from the bottom. It is noticeable a rise in the water surface just upwind of the hill and a dip of the water surface just downwind of the hill.

COMMENTS ON THE EXACT SOLUTIONS FOR THE WIND-DRIVEN CIRCULATION AROUND A SUBAQUEOUS POOL

Most cases presented in the figures from Fig. 27 to Fig. 34 correspond to pools twice as deep as the main water bodies. The only two exceptions are Fig. 29 and Fig. 31.

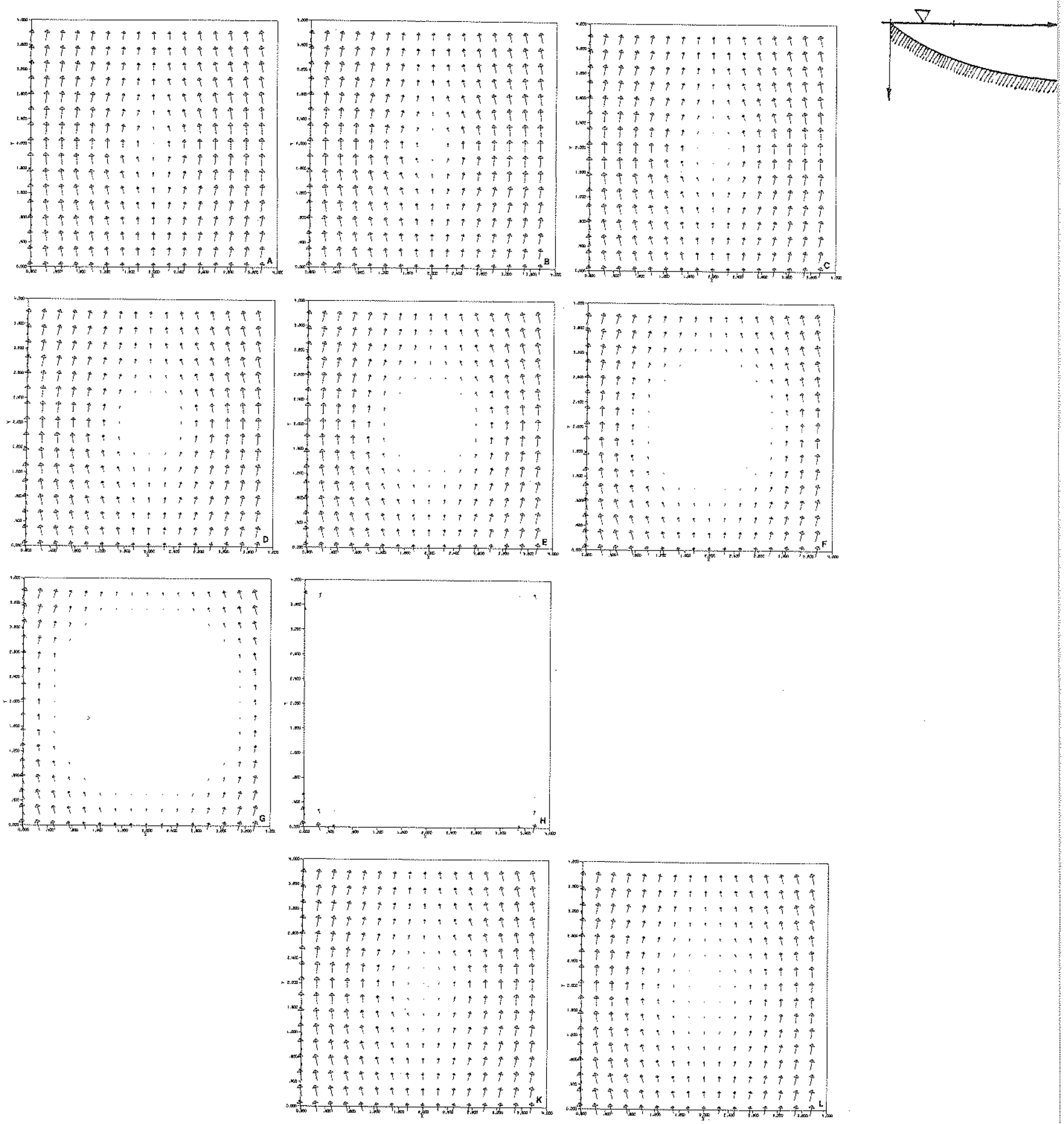
The most interesting feature of the case is constituted by the fact that all the water within the pool is in back flow mode, and so is part of the water just above the pool.

The pressure distribution over the free surface has not been illustrated by drawings since it is only slightly removed from a planar distribution growing in the direction of the wind.



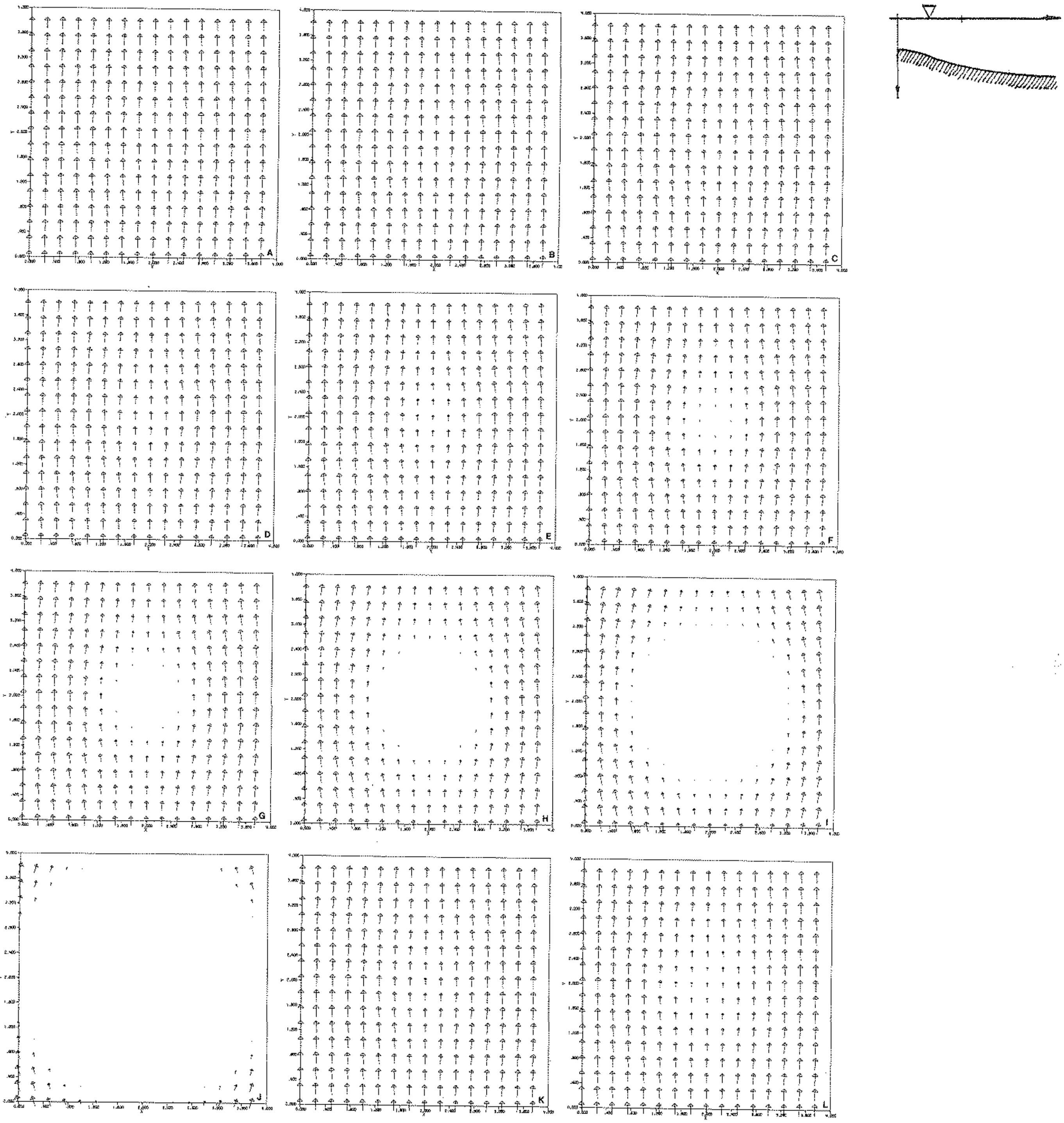
SUBAQUEOUS HILL BATHYMETRY: $h = 1 - \frac{.50}{1 + r}$

Fig. 12 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.



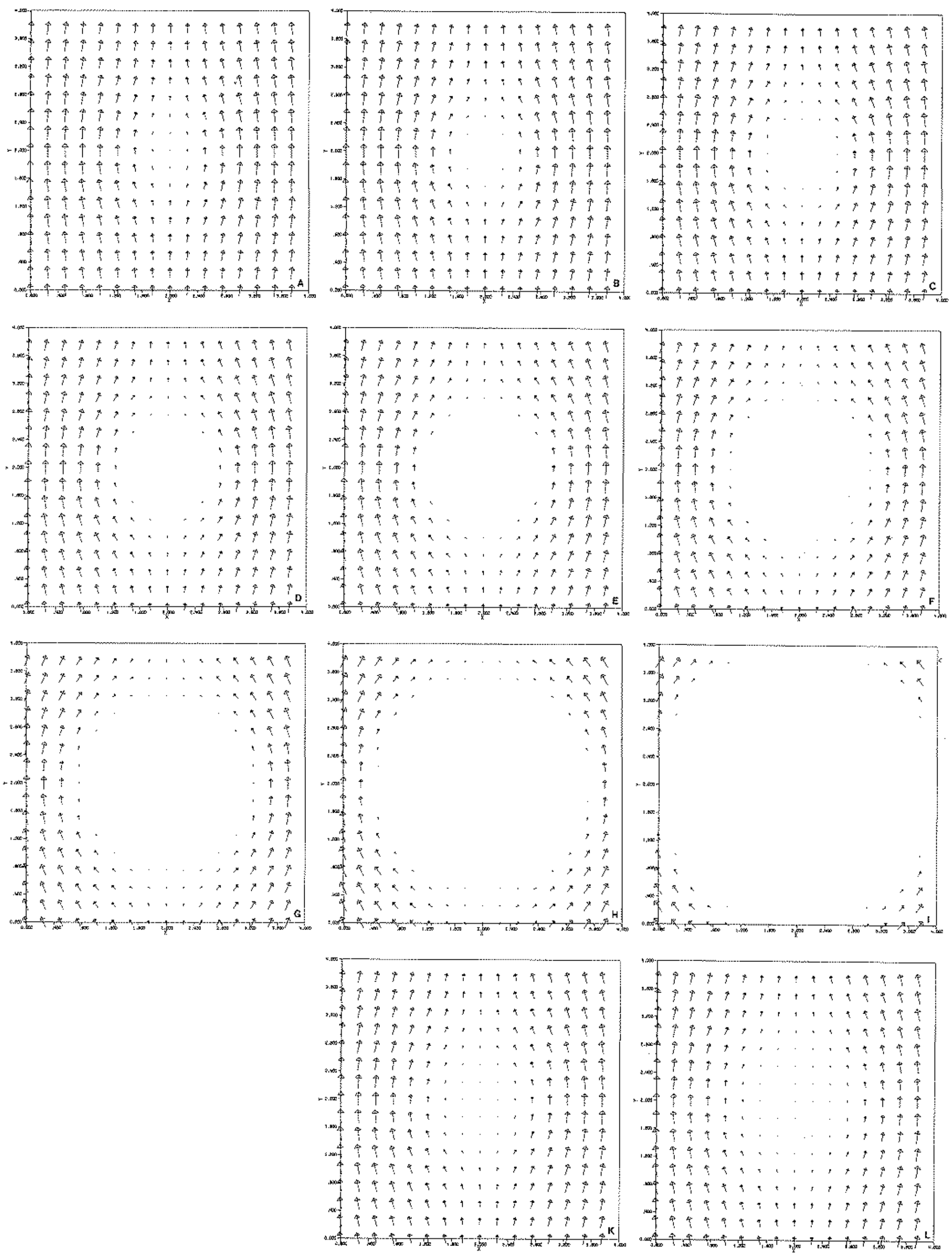
SUBAQUEOUS HILL BATHYMETRY: $h = 1 - \frac{.99}{1 + r}$

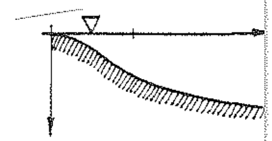
ig. 13 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.



SUBAQUEOUS HILL BATHYMETRY: $h = 1 - \dfrac{.50}{1 + r^2}$

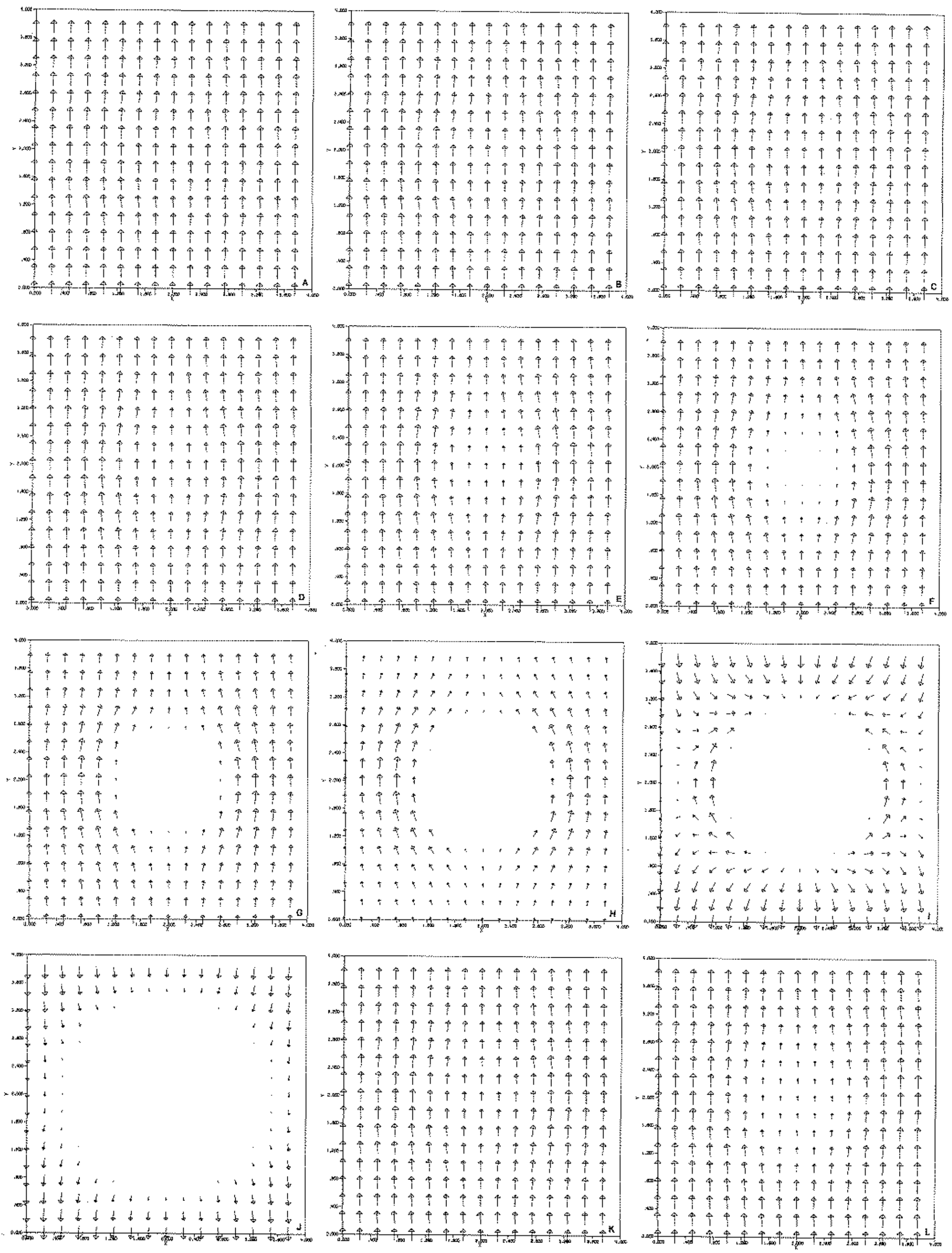
Fig. 14 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.

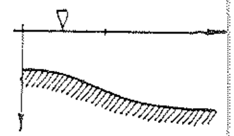




SUBAQUEOUS HILL BATHYMETRY: $h = 1 - \dfrac{.99}{1 + r^2}$

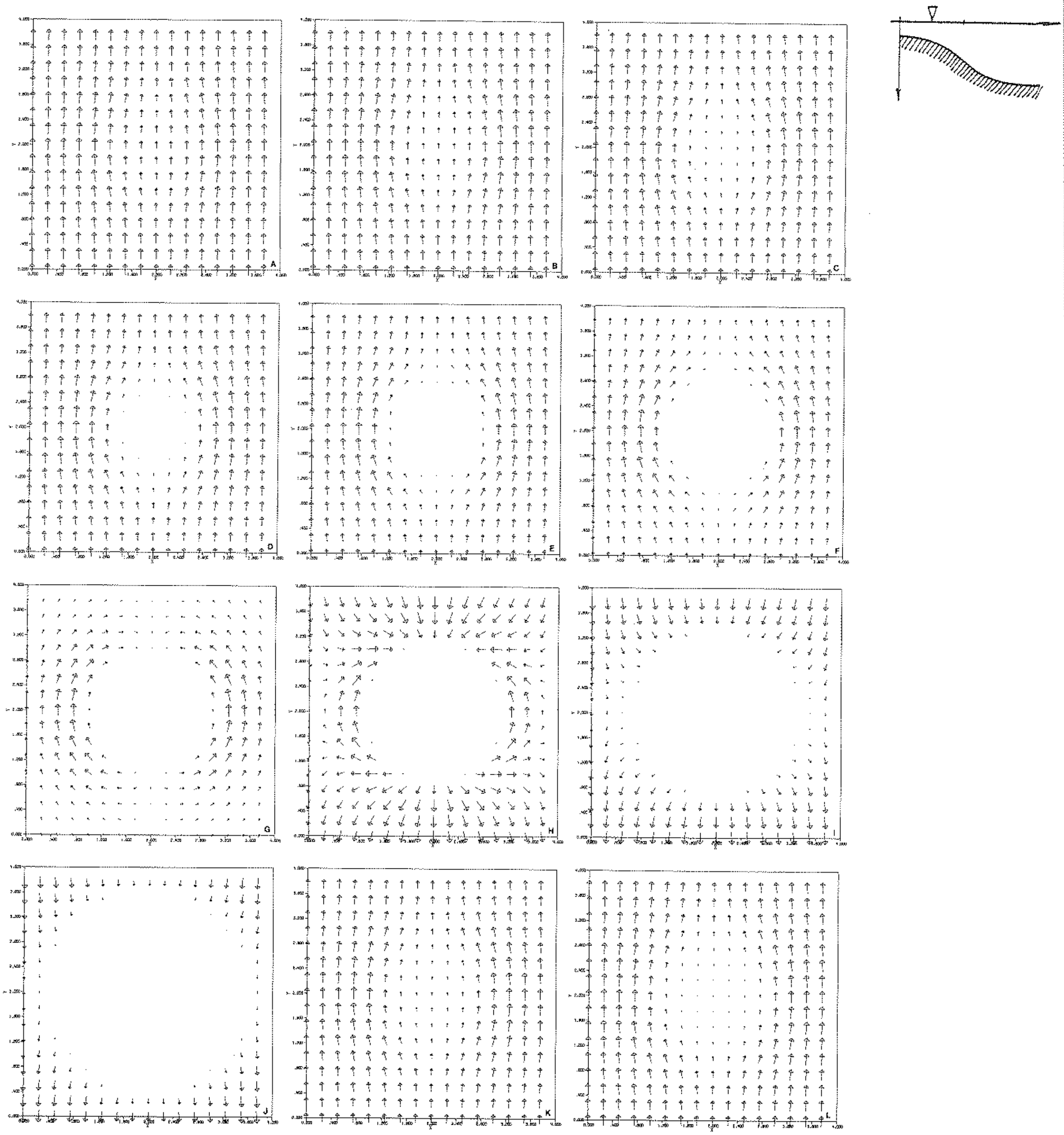
Fig. 15 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.





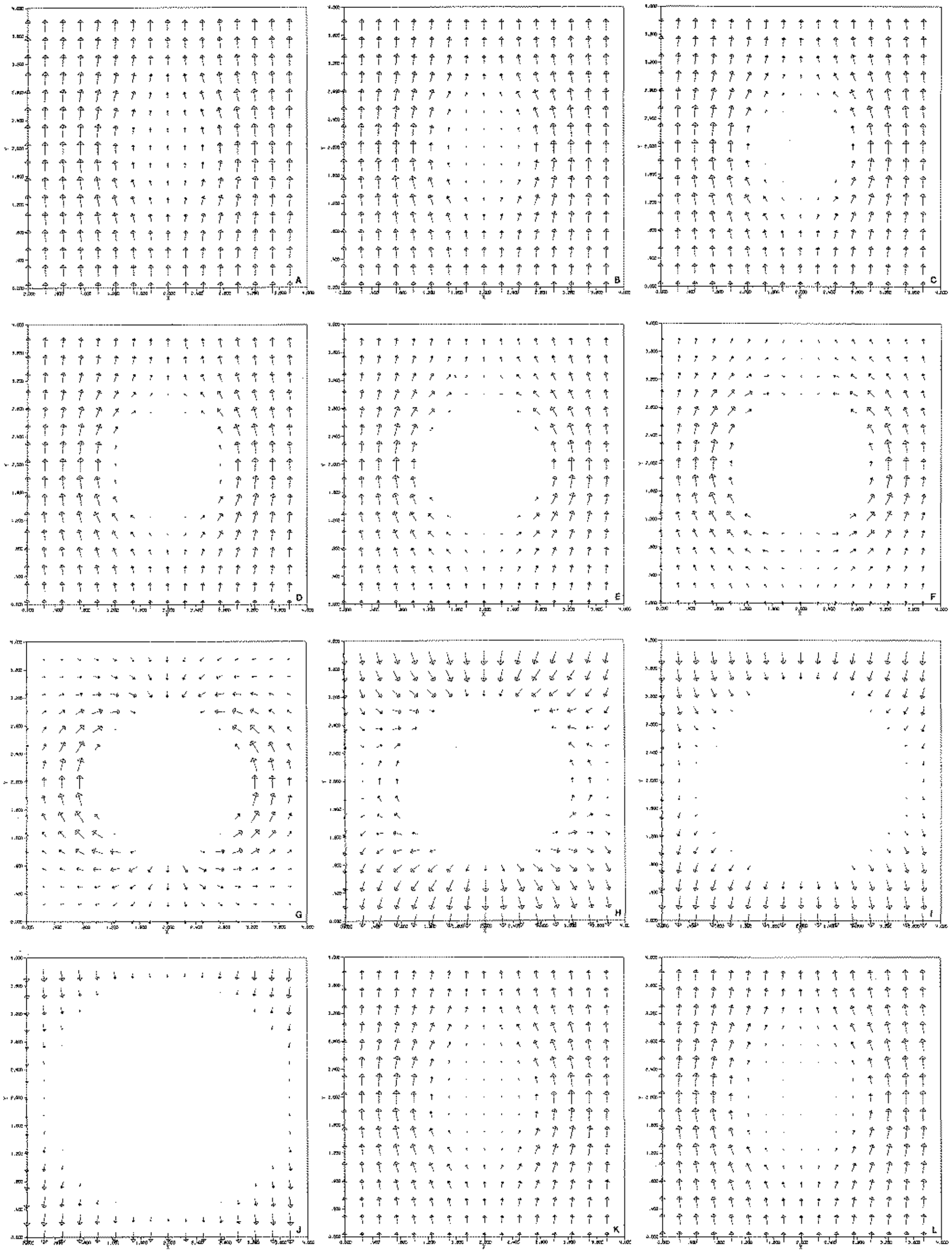
$$\text{SUBAQUEOUS HILL BATHYMETRY:} \quad h = 1 - \frac{.50}{1 + r^4}$$

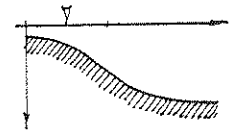
Fig. 16 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.



SUBAQUEOUS HILL BATHYMETRY: $h = 1 - \frac{.75}{1 + r^4}$

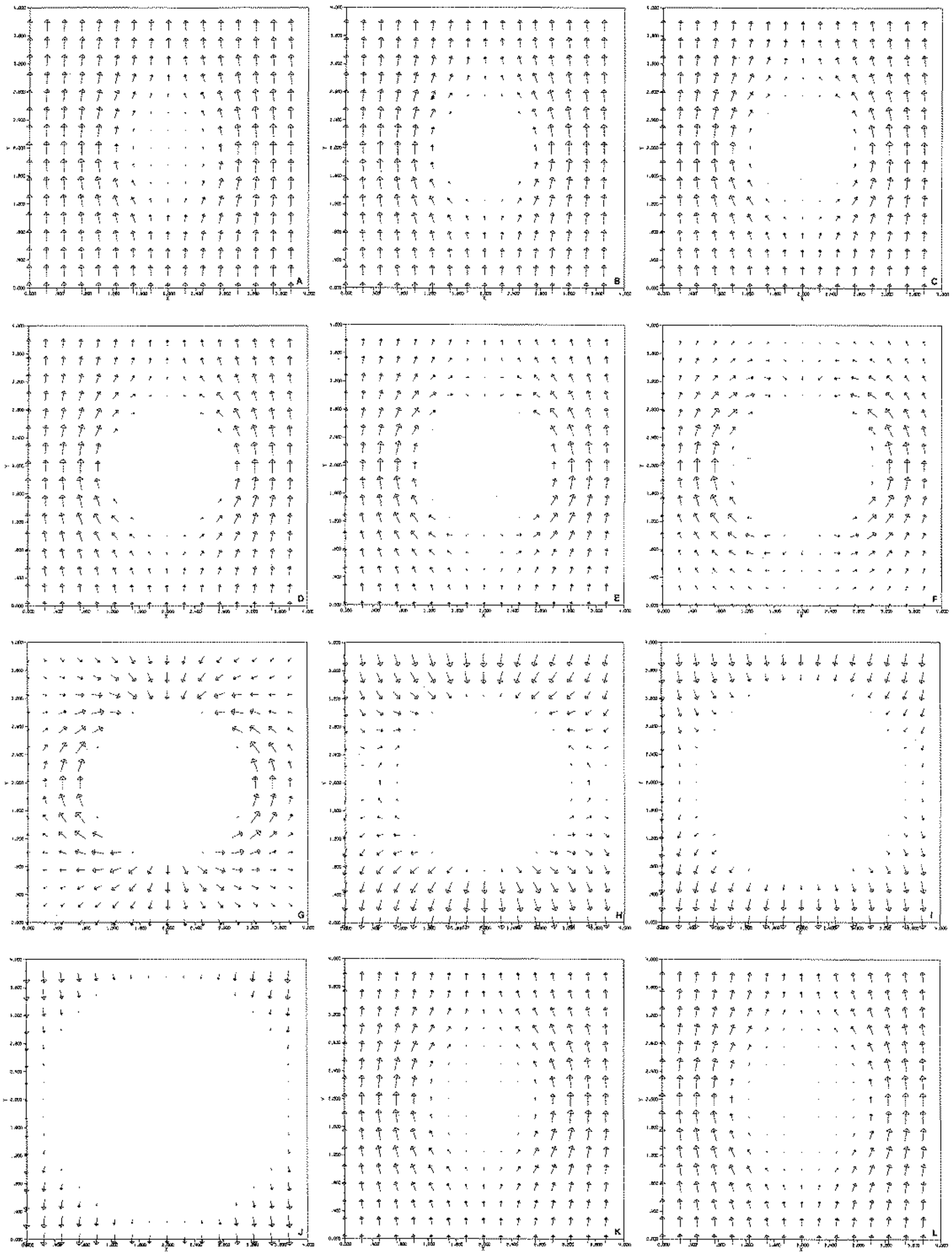
Fig. 17 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.

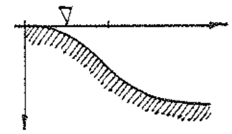




SUBAQUEOUS HILL BATHYMETRY: $h = 1 - \frac{.875}{1 + r^4}$

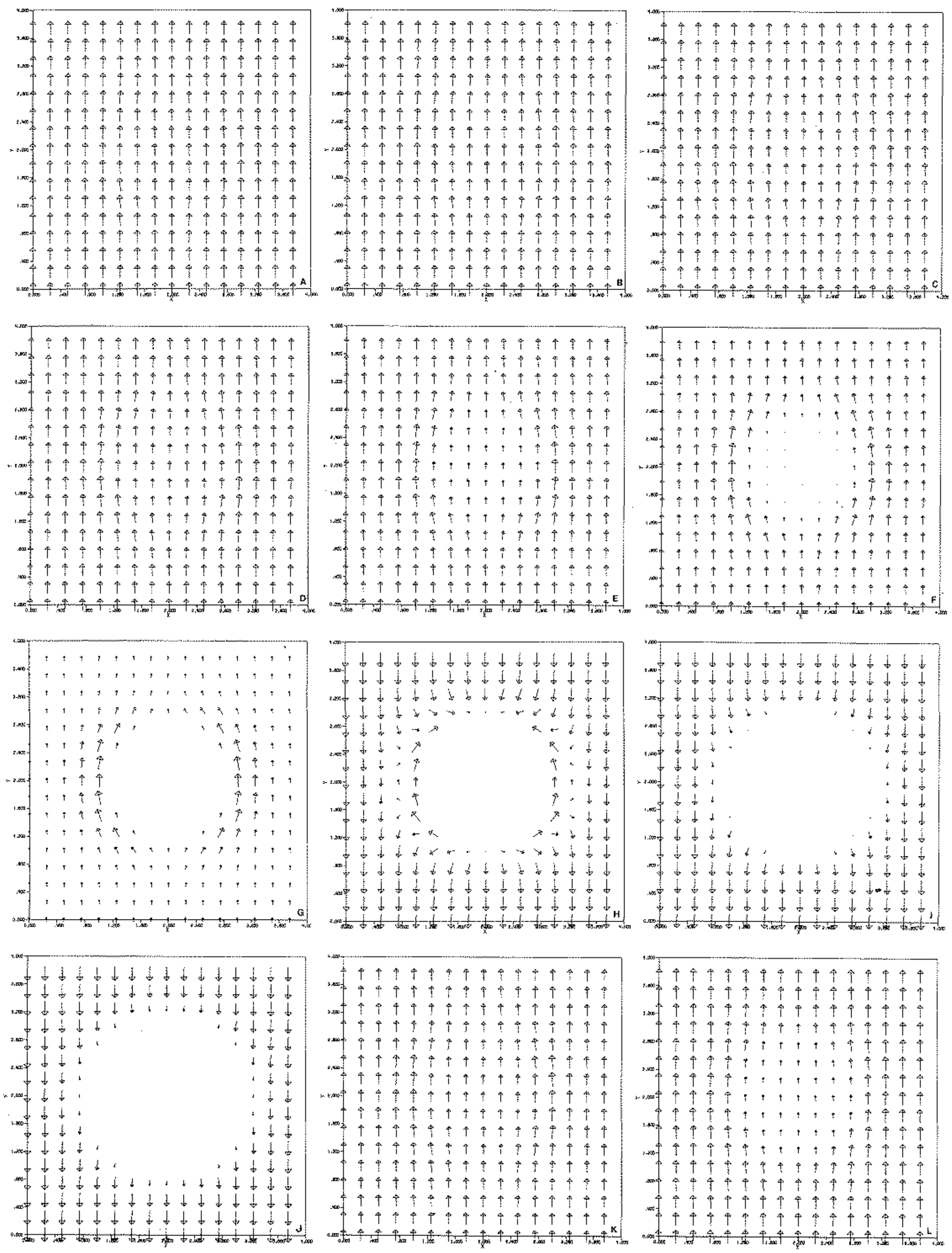
Fig. 18 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.

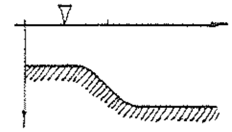




SUBAQUEOUS HILL BATHYMETRY: $h = 1 - \frac{.99}{1 + r^4}$

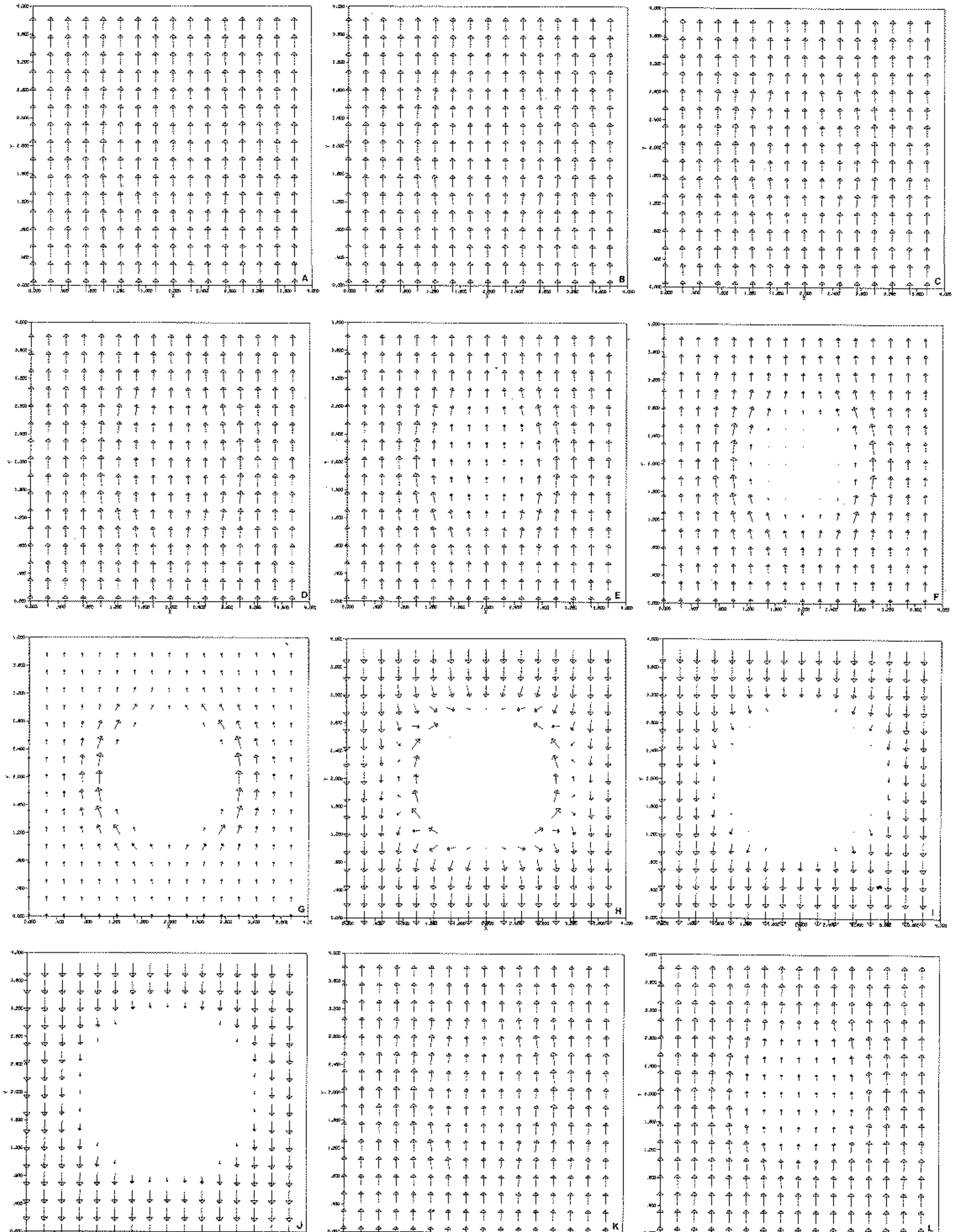
Fig. 19 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.

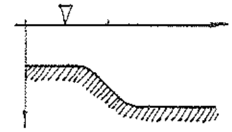




SUBAQUEOUS HILL BATHYMETRY: $h = 1 - \frac{.50}{1 + r^8}$

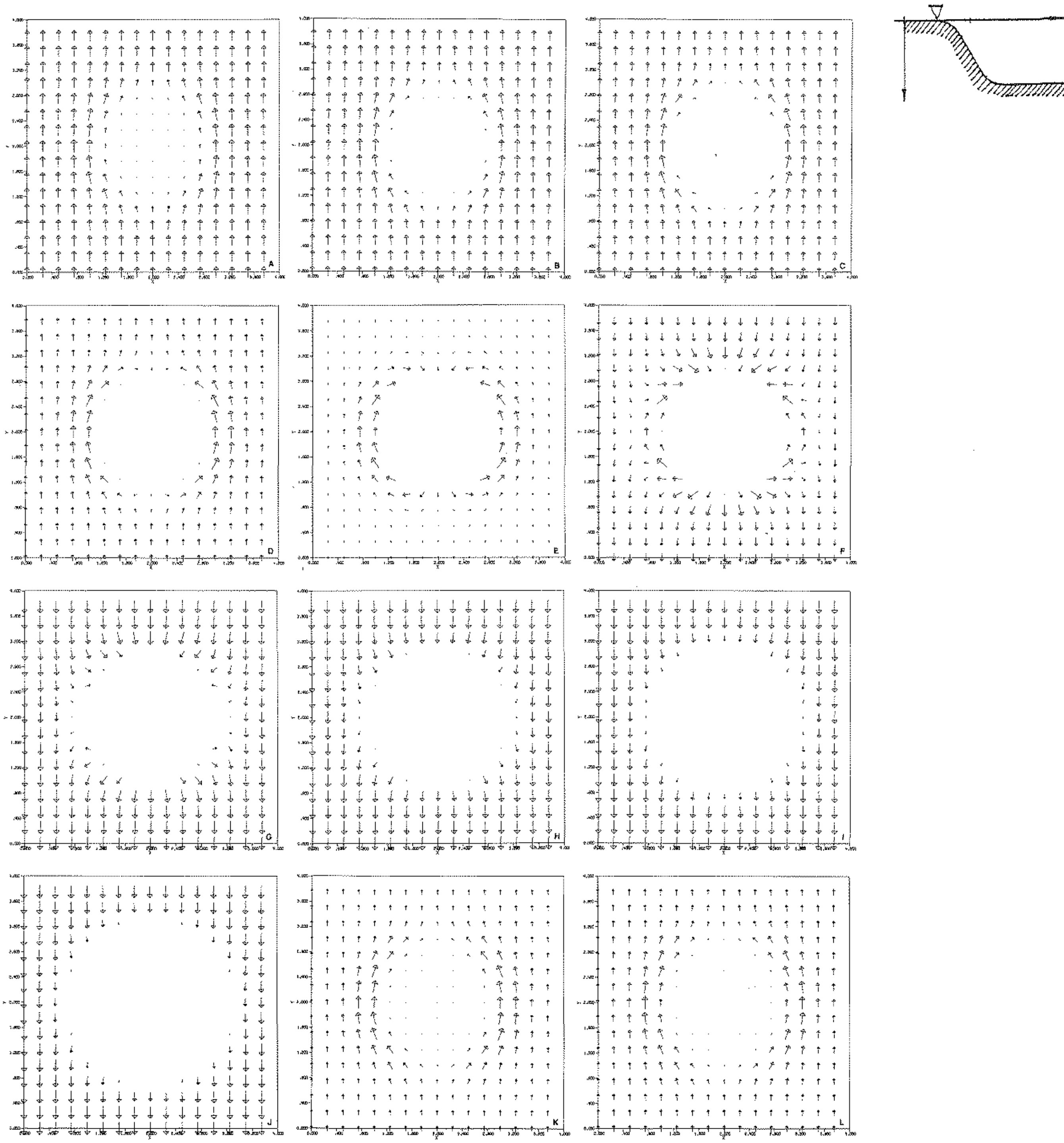
Fig. 20 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.





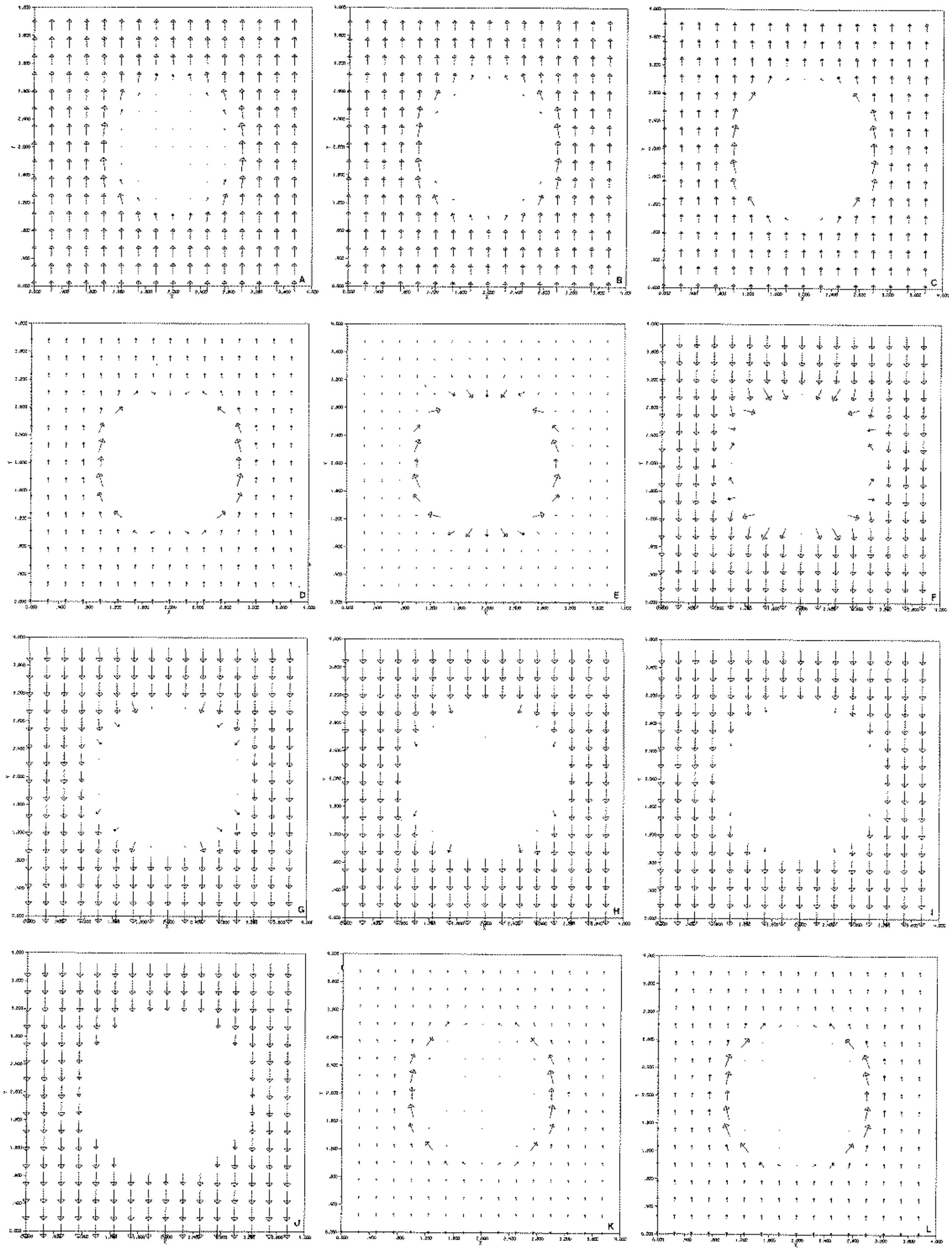
SUBAQUEOUS HILL BATHYMETRY: $h = 1 - \dfrac{.50}{1 + r^8}$

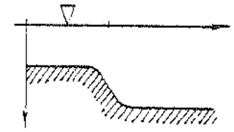
Fig. 20 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from $z = 0$. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.



$$\text{SUBAQUEOUS HILL BATHYMETRY:} \quad h = 1 - \frac{.99}{1 + r^8}$$

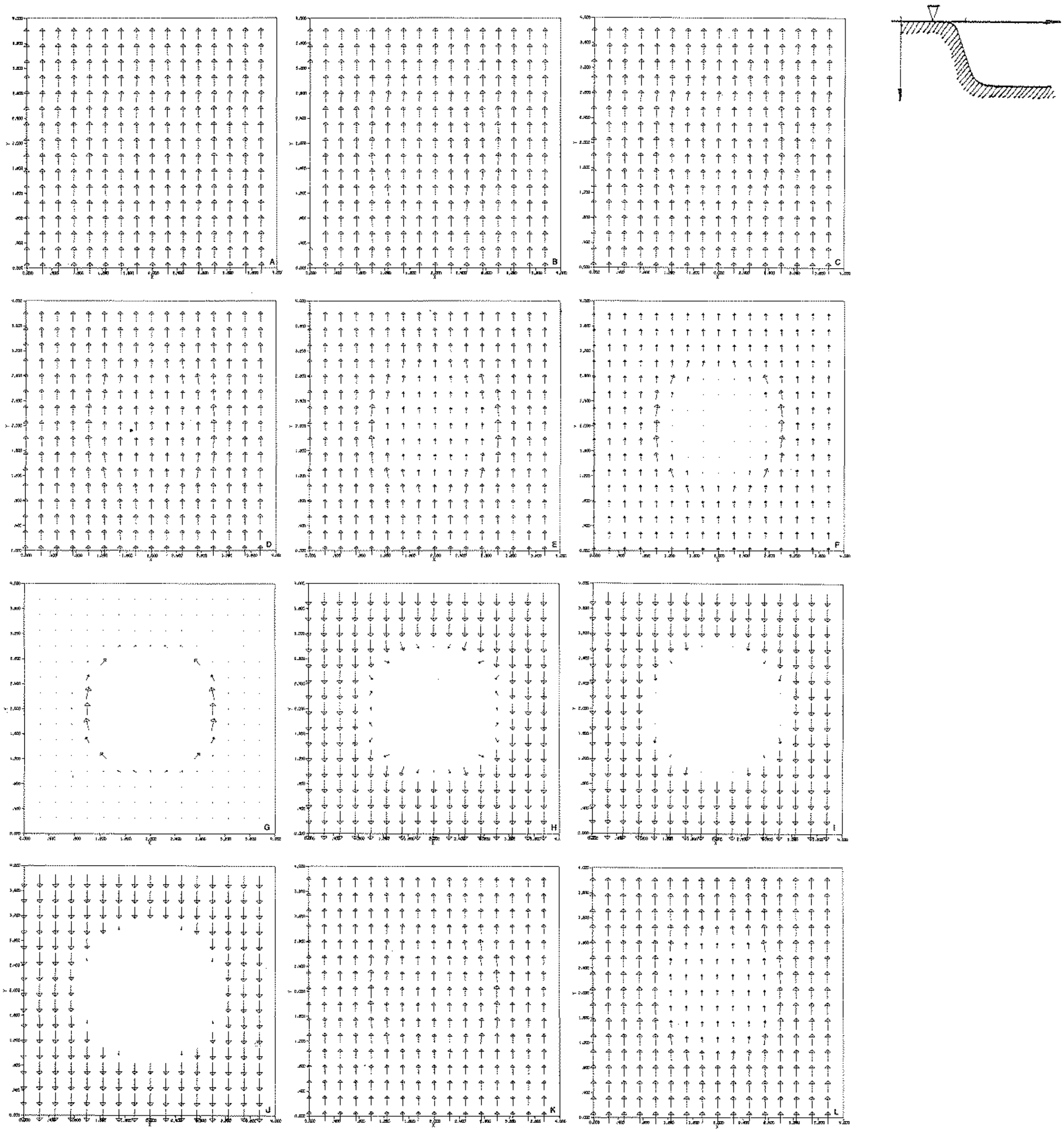
Fig. 21 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.





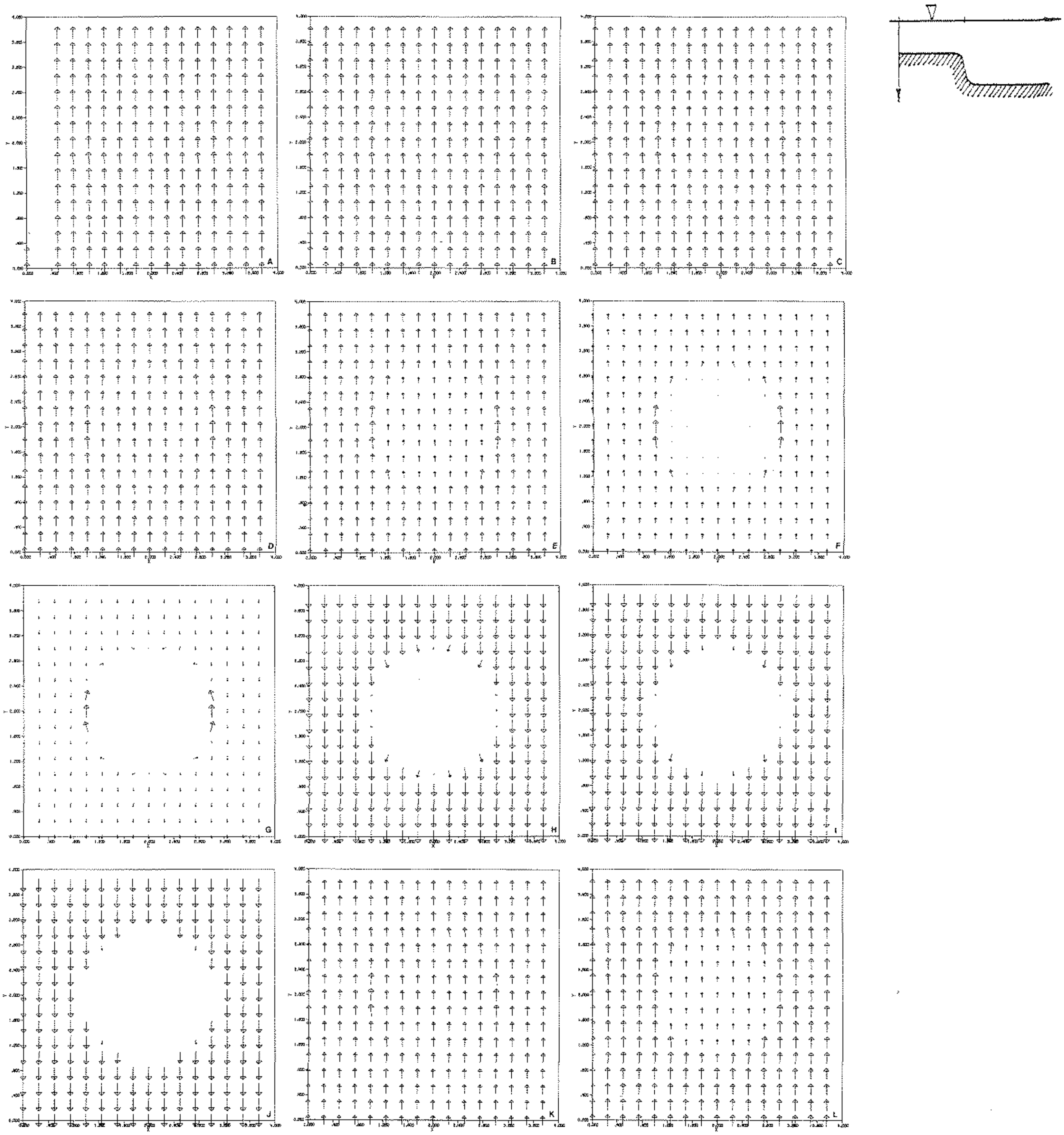
SUBAQUEOUS HILL BATHYMETRY: $h = 1 - \dfrac{.99}{1 + r^{16}}$

ig. 22 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the hcrizontal specific flowrates.



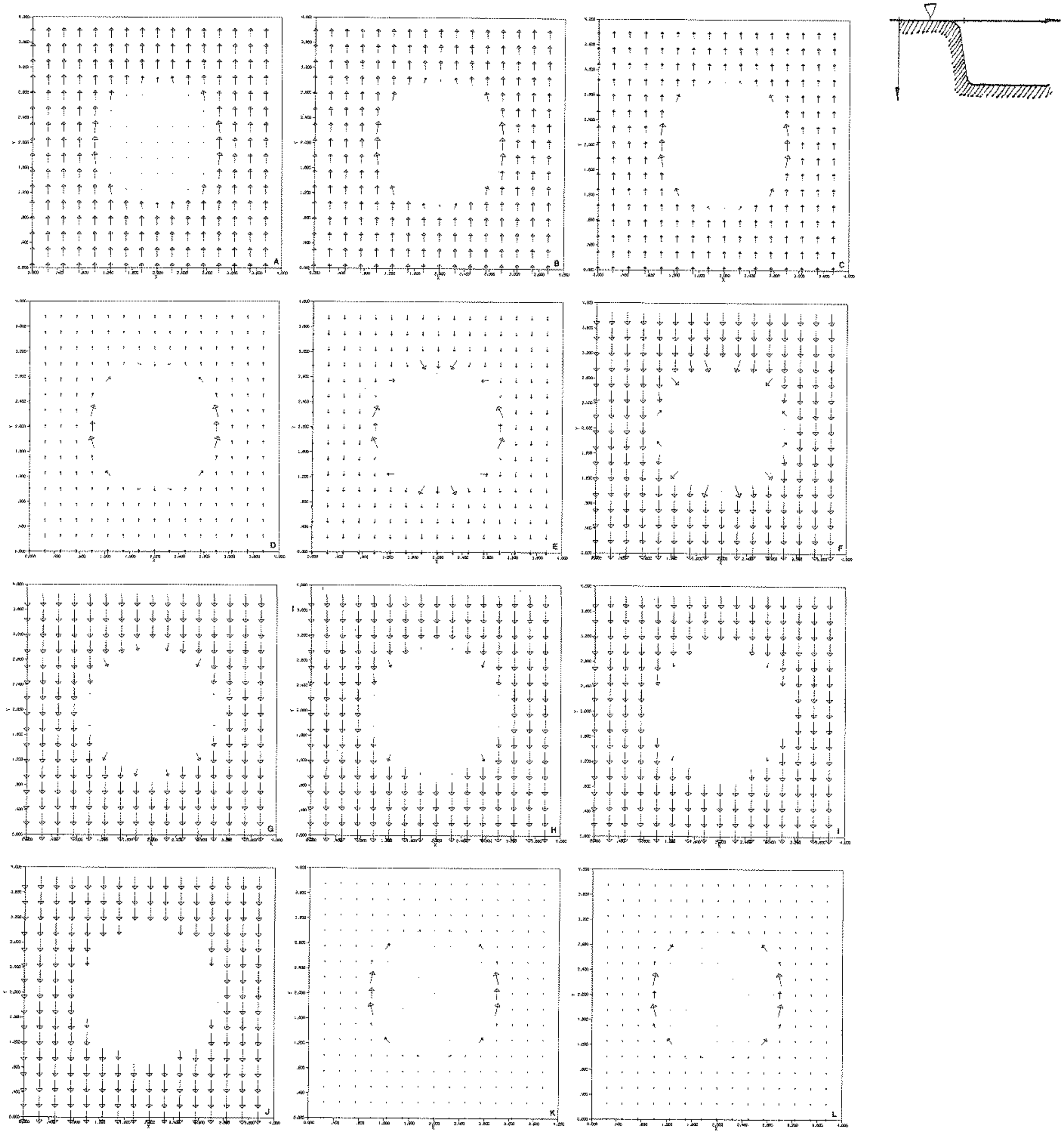
SUBAQUEOUS HILL BATHYMETRY: $h = 1 - \dfrac{.50}{1 + r^{16}}$

Fig. 23 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.



SUBAQUEOUS HILL BATHYMETRY: $h = 1 - \dfrac{.50}{1 + r^{32}}$

Fig. 24 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.



SUBAQUEOUS HILL BATHYMETRY: $h = 1 - \frac{.99}{1 + r^{32}}$

ig. 25 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.

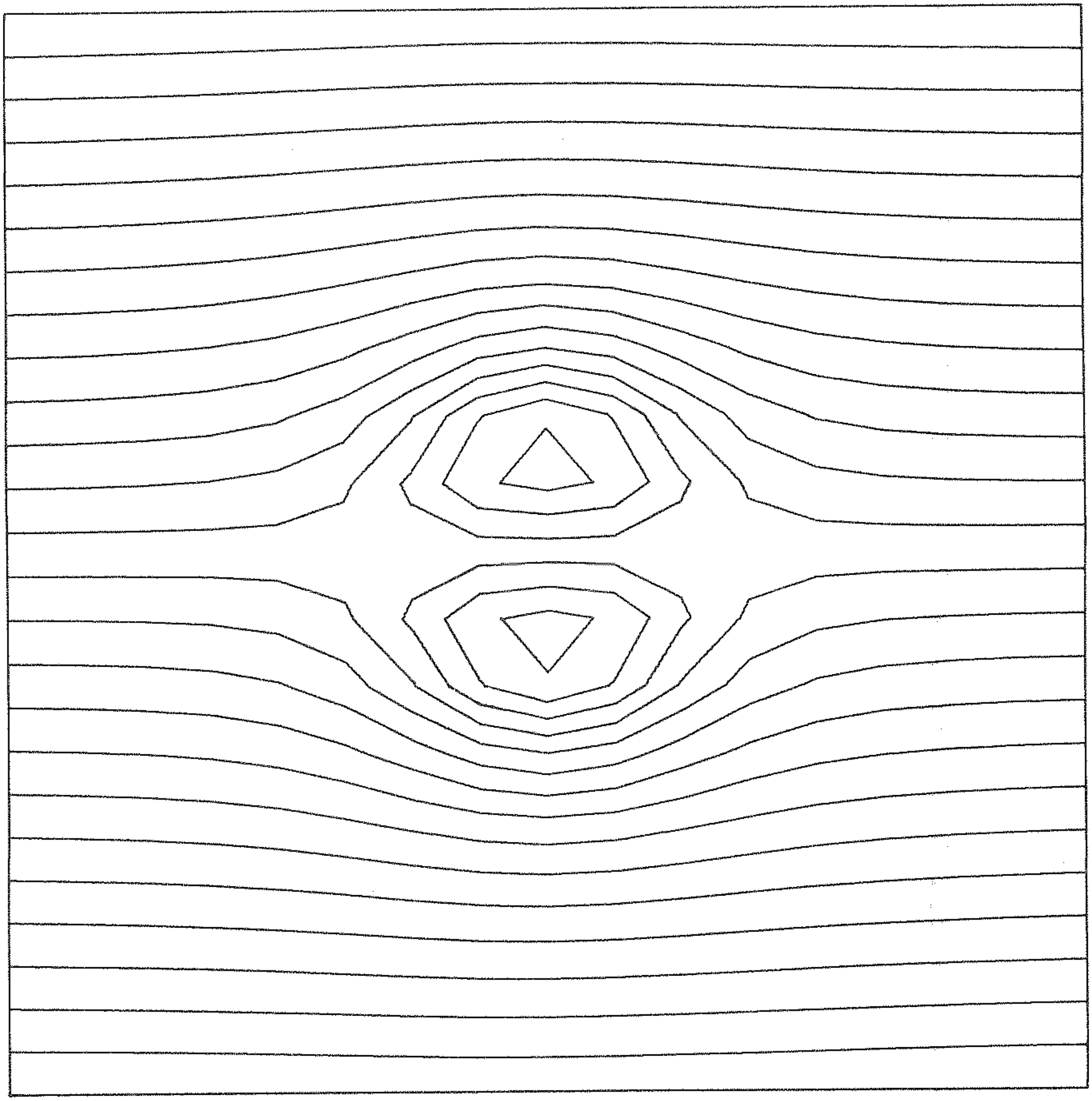
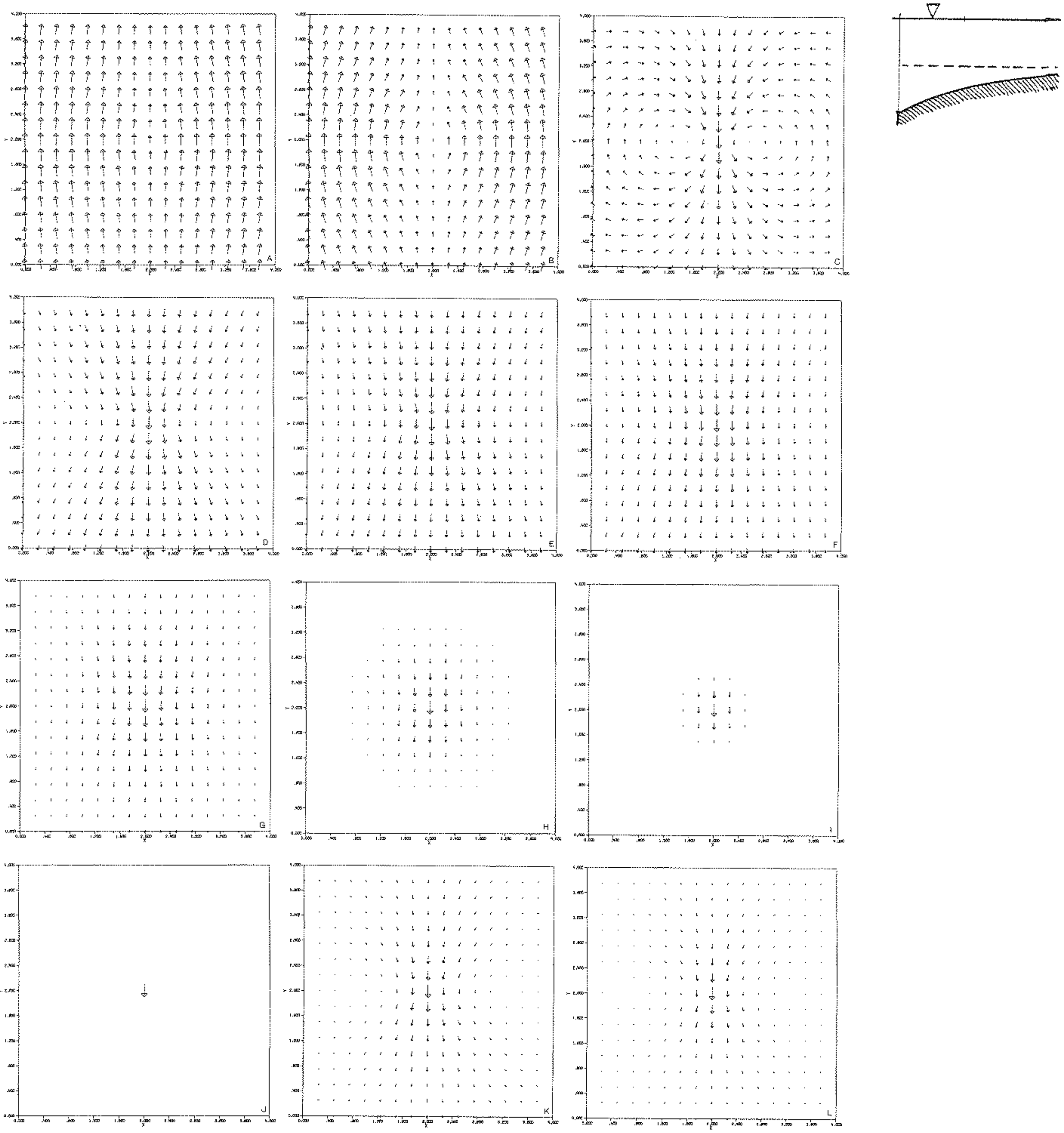
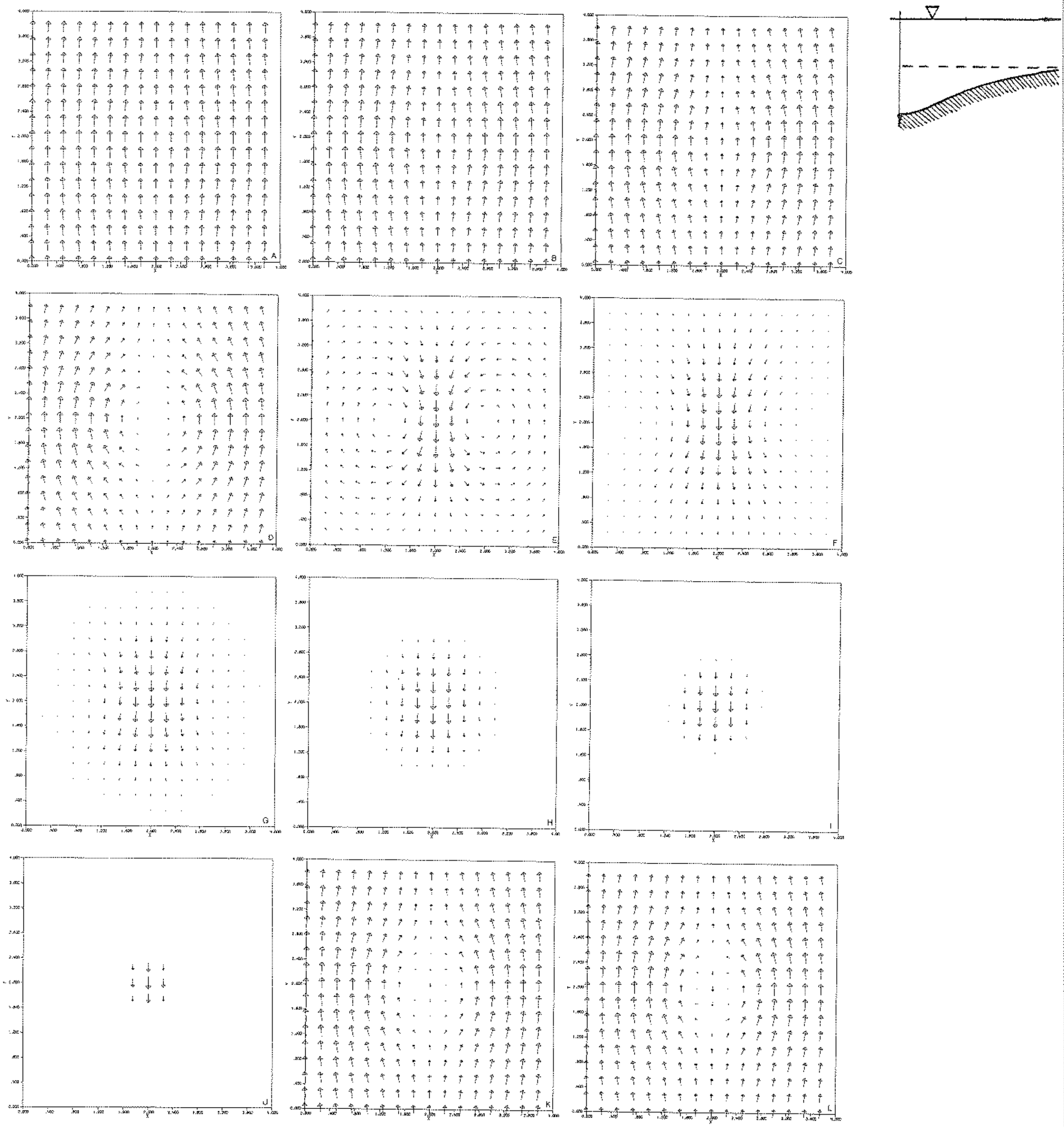


Fig. 18 - Pressure distribution abroad the lake for the bathymetry of Fig. . The lines are isobaric lines drawn for equal increment of pressure. They may be considered the contour lines for the lake setup.



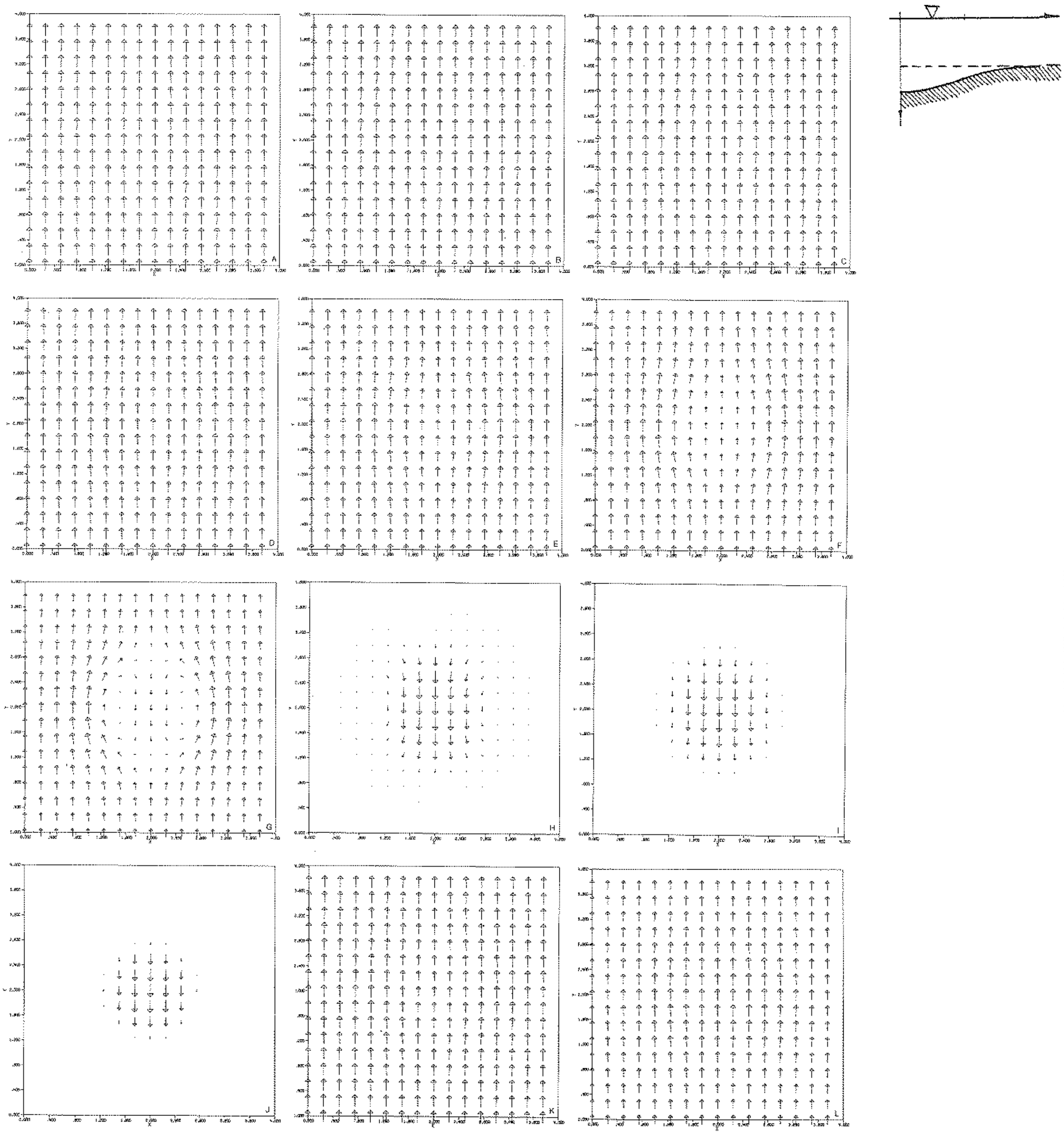
SUBAQUEOUS POOL BATHYMETRY: $H = 1 + \frac{1}{1 + r}$

Fig. 27 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.



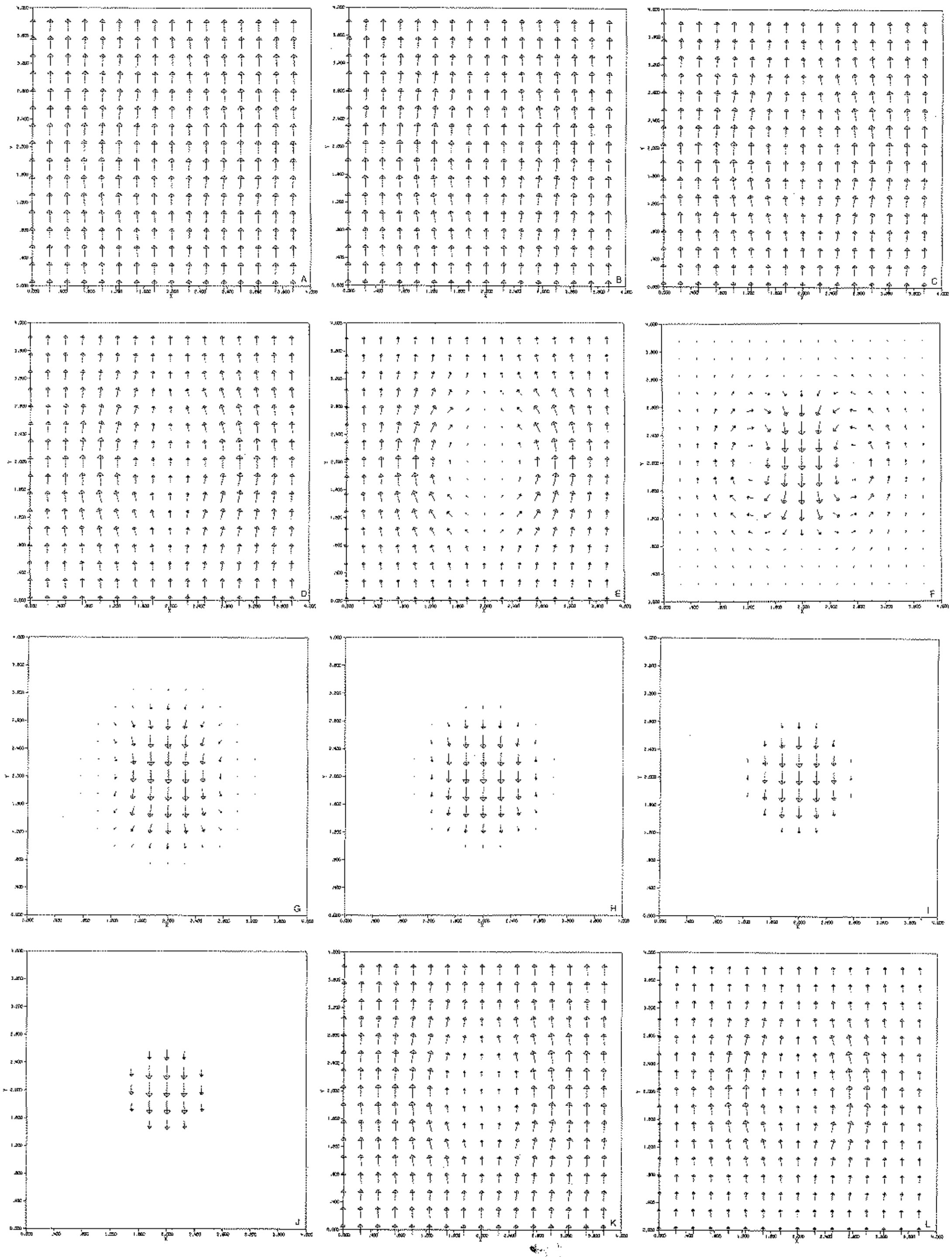
SUBAQUEOUS POOL BATHYMETRY: $h = 1 + \frac{1}{1 + r^2}$

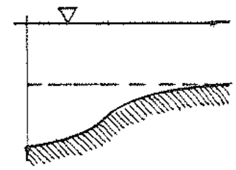
Fig. 28 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from $z = 0$. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.



SUBAQUEOUS POOL BATHYMETRY: $h = 1 + \dfrac{.50}{1 + r^4}$

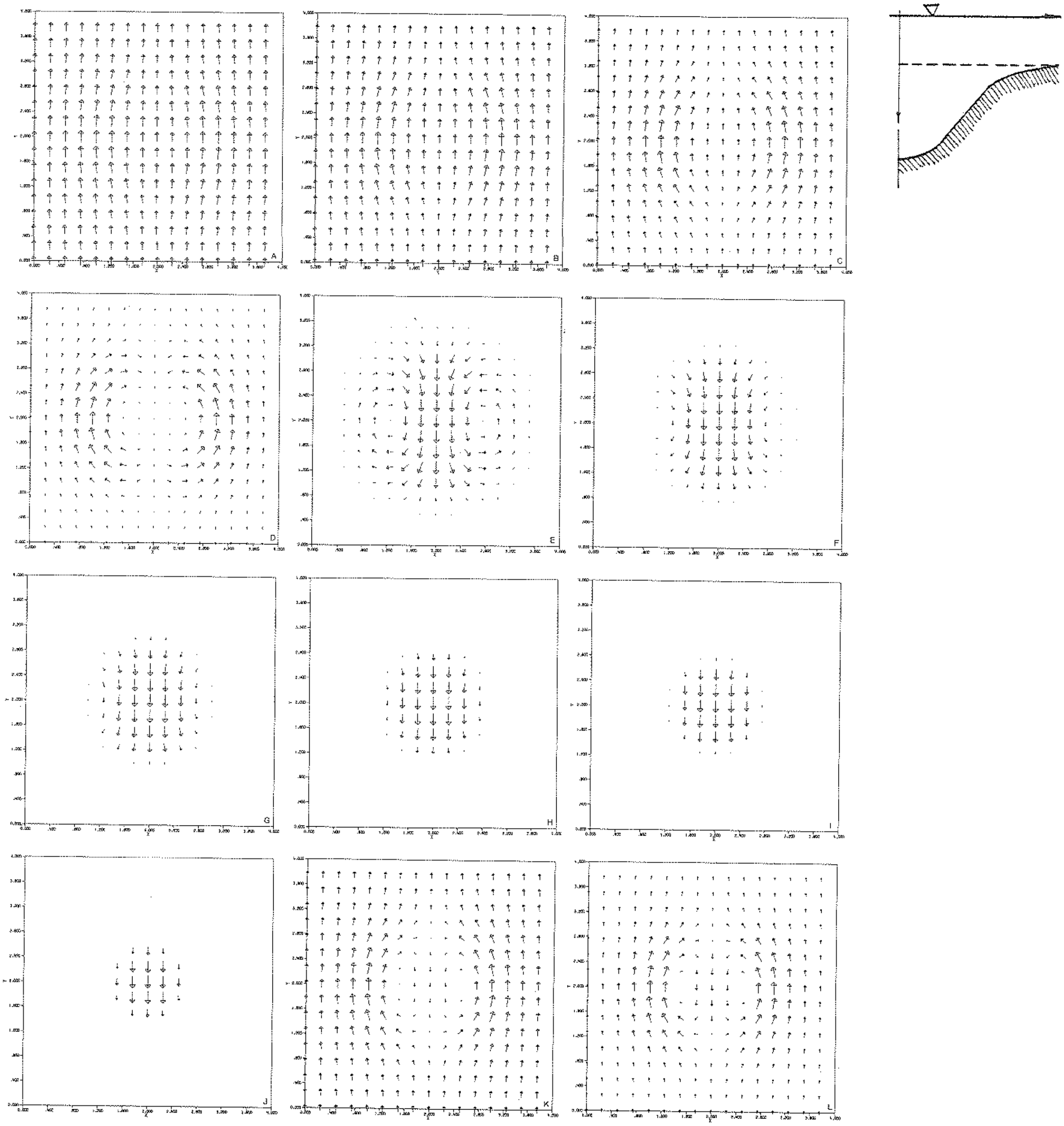
Fig. 29 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.





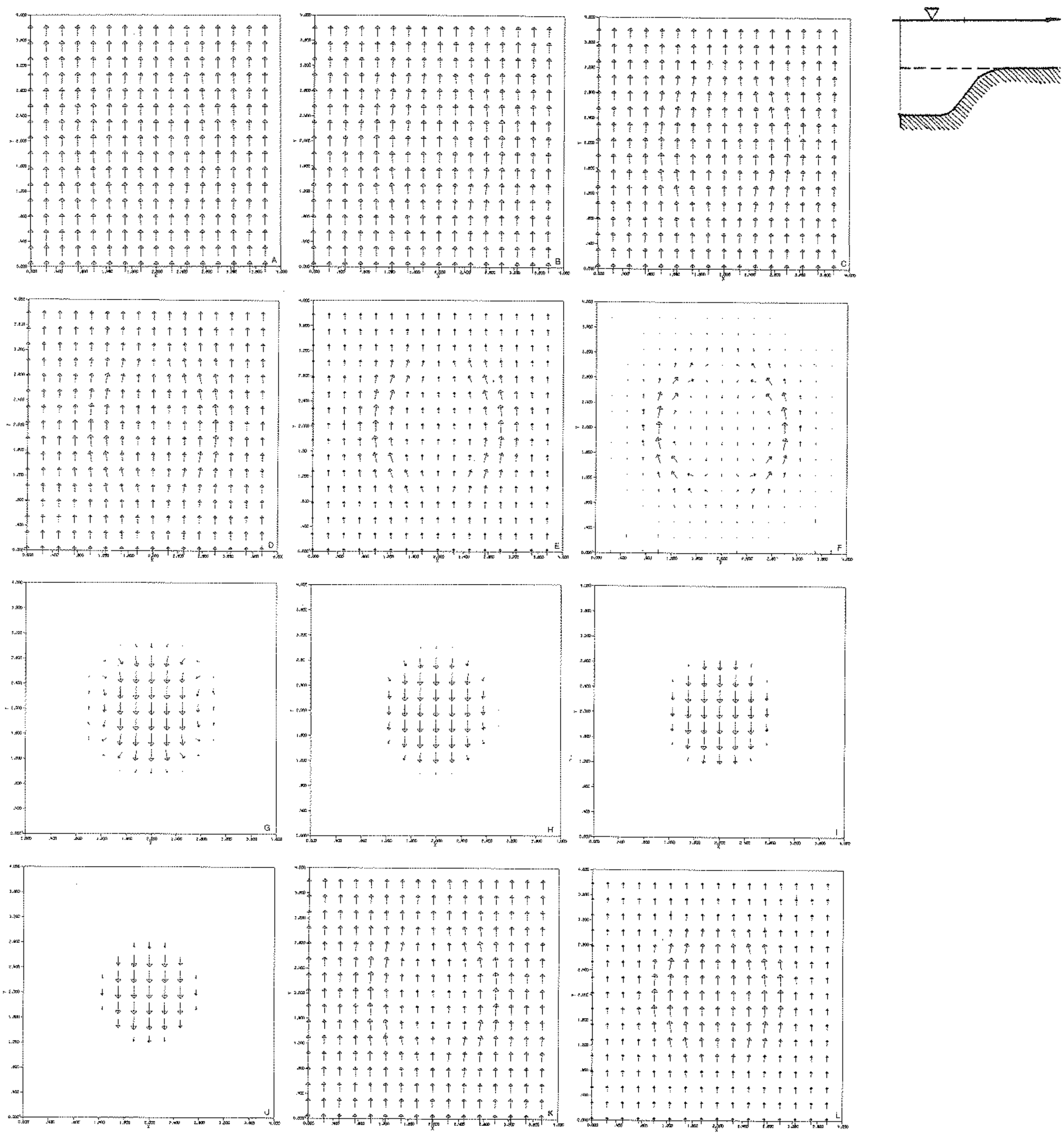
SUBAQUEOUS POOL BATHYMETRY: $h = 1 + \dfrac{1}{1 + r^4}$

Fig. 30 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.



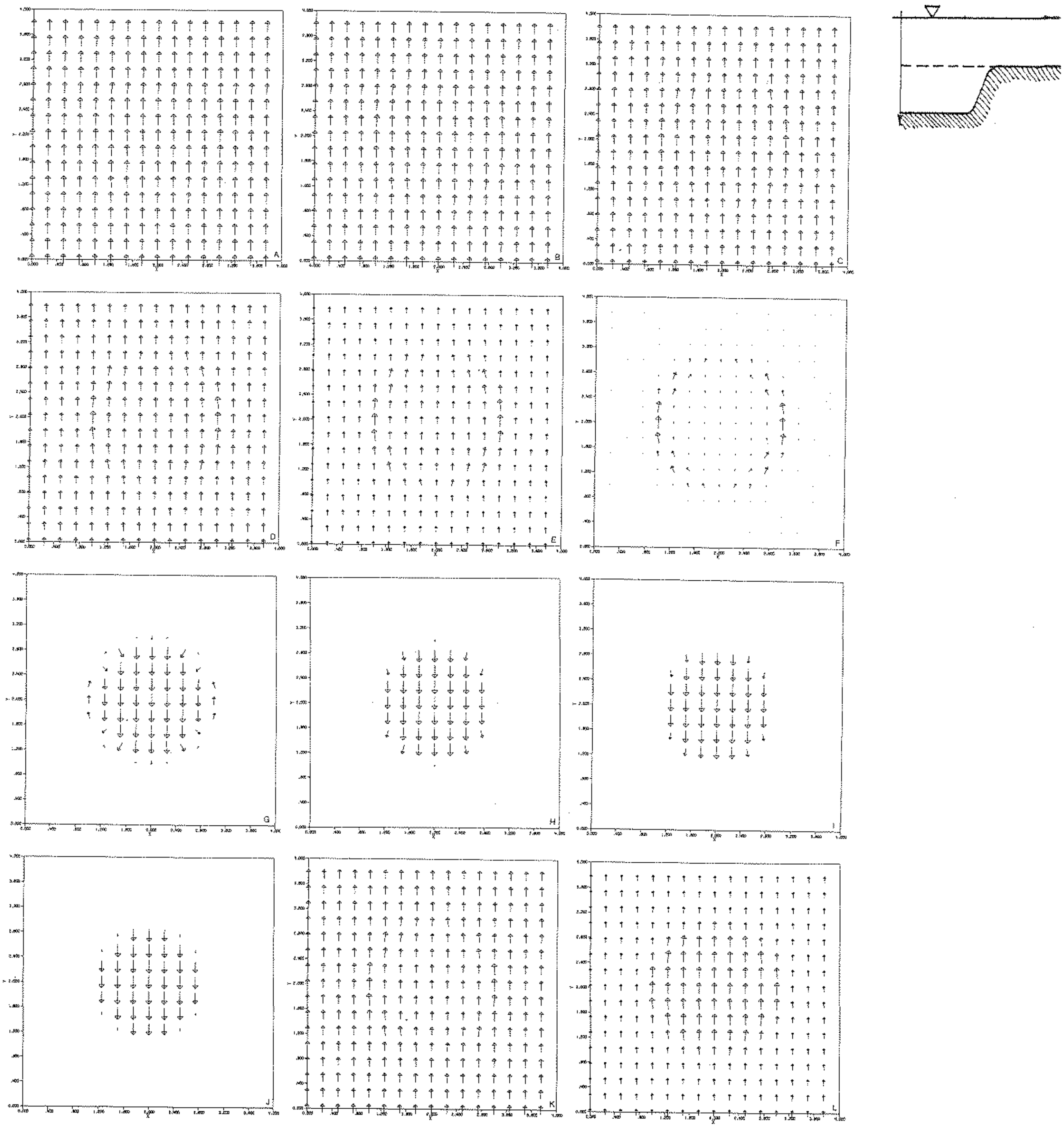
SUBAQUEOUS POOL BATHYMETRY: $h = 1 + \dfrac{2}{1 + r^4}$

Fig. 31 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.



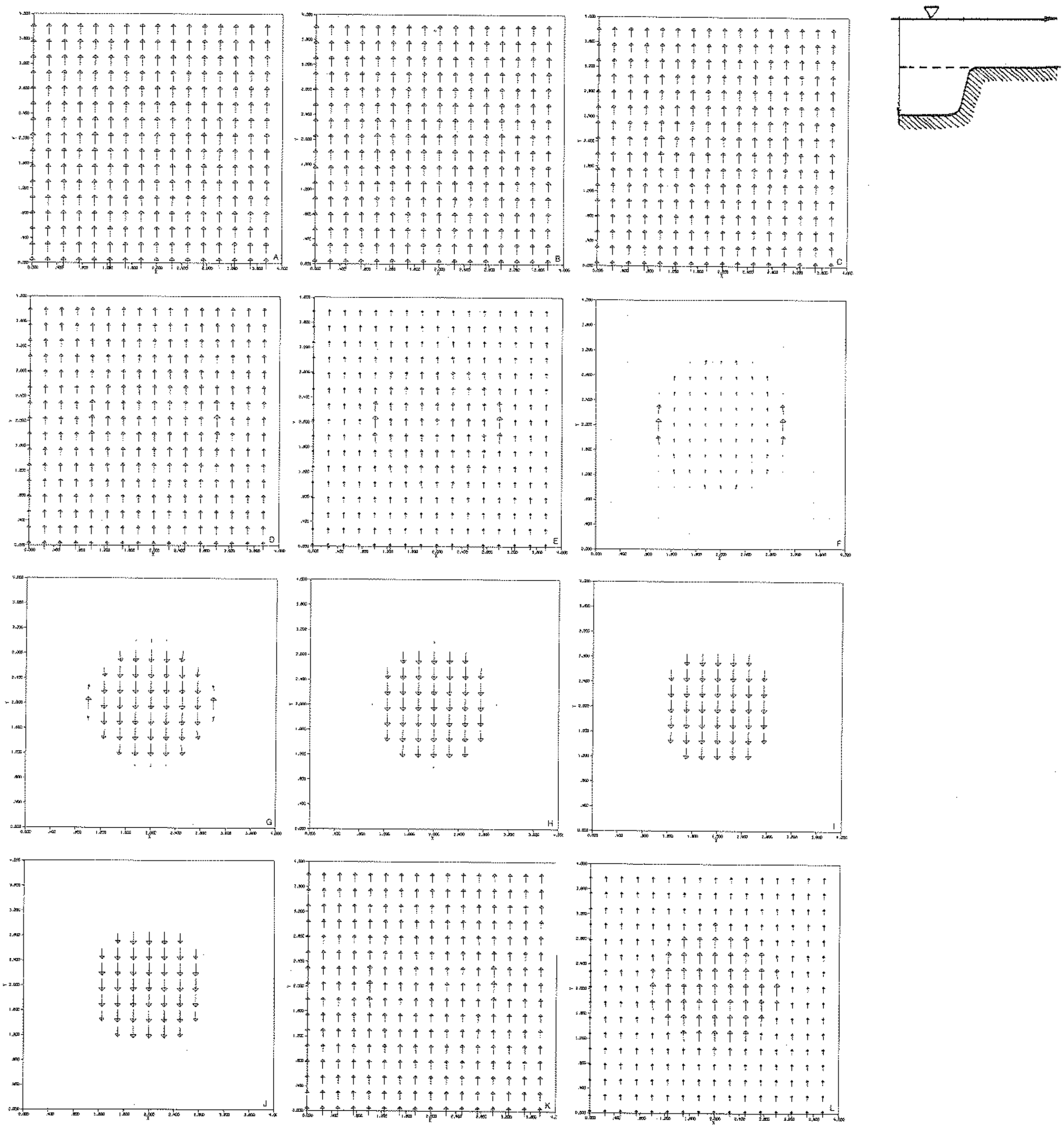
SUBAQUEOUS POOL BATHYMETRY: $h = 1 + \dfrac{1}{1 + r^8}$

Fig. 32 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.



SUBAQUEOUS POOL BATHYMETRY: $h = 1 + \dfrac{1}{1 + r^{16}}$

Fig. 33 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from $z = 0$. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.



SUBAQUEOUS POOL BATHYMETRY: $h = 1 + \dfrac{1}{1 + r^{32}}$

Fig. 34 - Velocity field with constant surface shear. Frames A to J present ten successively deeper equidistant horizontal cuts, starting from z = 0. Frame K presents the depth averaged horizontal velocity field. Frame L presents the horizontal specific flowrates.

CONCLUSIONS

The research presented in this report has two major points of interest. Firstly it presents a method for the solution of some mathematical problems (not necessarily of the lake circulation type). The method is based on a device that relaxes a boundary condition, and constrains the solution type.

Other applications of the method are very likely possible in other fluid mechanical or non-fluid mechanical problems.

The second point, related to the usefulness of the exact solutions presented, has a twofold direction. On one hand these exact solutions are an excellent testing bench for numerical schemes of computational fluid mechanics; on the other hand, due to the reasonable form of the bathymetry, they can constitute the basis for other studies related to lake circulation. An application of this last statement is being worked out now by the author in the study of sediment transport in lakes.

The abundance of graphical representation, in lieu of countless pages of codes and numerical table, has been given in order to preserve some comprehensive illustration of the solution, which would not be evident by the form of the equations.

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APPENDIX A

RELATIONSHIP BETWEEN VERTICAL VELOCITY COMPONENT AND PRESSURE FIELD

The continuity equation (10) is first differentiated two times with respect to z to yield

$$\frac{\partial}{\partial x}\frac{\partial^2 u}{\partial z^2} + \frac{\partial}{\partial y}\frac{\partial^2 v}{\partial z^2} + \frac{1}{\lambda}\frac{\partial^3 w}{\partial z^3} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (A\text{-}1)$$

Substituting (8) and (9) into (A-1) we get

$$\lambda\frac{\partial^2 p_o}{\partial x^2} + \lambda\frac{\partial^2 p_o}{\partial y^2} + \frac{1}{\lambda}\frac{\partial^3 w}{\partial z^3} = 0$$

or

$$\frac{\partial^3 w}{\partial z^3} = -\lambda^2\nabla^2 p_o \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A\text{-}2)$$

A first integration of (A-2) yields

$$\frac{\partial^2 w}{\partial z^2} = -\lambda^2\nabla^2 p_o z + M$$

and a second integration (from -h to z with the use of the boundary condition $\left.\frac{\partial w}{\partial z}\right|_{-h} = 0$)

$$\frac{\partial w}{\partial z} = -\lambda^2\nabla^2 p_o\,\frac{z^2-h^2}{2} + M(z+h)$$

A further integration from 0 to z yields

$$w = -\frac{\lambda^2}{2}\,\nabla^2 p_o\left(\frac{z^3}{3} - h^2 z\right) + M\left(\frac{z^2}{2} + hz\right) \quad \ldots\ldots\ldots\ldots\ldots (A\text{-}3)$$

The boundary condition w(-h) = 0 applied to (A-3) yields

$$M = -\frac{2}{3}\,\lambda^2 h\nabla^2 p_o \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A\text{-}4)$$

which, once substituted into (A-3), produces

$$w = -\frac{\lambda^2}{6}\nabla^2 p_o z(z+h)^2. \quad \text{(A-5)}$$

Since in the present paper $p_o = yG(h)$, we will derive an expression for $\nabla^2 p_o$ in terms of G and h:

$$\frac{\partial p_o}{\partial x} = yG' \frac{\partial h}{\partial r}\frac{x}{r} = \frac{xy}{r}\frac{\partial h}{\partial r} G'. \quad \text{(A-6)}$$

$$\frac{\partial p_o}{\partial y} = yG' \frac{\partial h}{\partial r}\frac{y}{r} + G = \frac{y^2}{r}\frac{\partial h}{\partial r} G' + G \quad \text{(A-7)}$$

$$\frac{\partial^2 p_o}{\partial x^2} = \frac{y^3}{r^3}\frac{\partial h}{\partial r} G' + \frac{x^2 y}{r^2}\frac{\partial^2 h}{\partial r^2} G' + \frac{x^2 y}{r^2}\left(\frac{\partial h}{\partial r}\right)^2 G''$$

$$\frac{\partial^2 p_o}{\partial y^2} = -\frac{y^3}{r^3}\frac{\partial h}{\partial r} G' + \frac{y^3}{r^2}\frac{\partial^2 h}{\partial r^2} G' + \frac{y^3}{r^2}\left(\frac{\partial h}{\partial r}\right)^2 G'' + 3\frac{y}{r} G' \frac{\partial h}{\partial r}$$

and

$$\nabla^2 p_o = y\frac{\partial^2 h}{\partial r^2} G' + y\left(\frac{\partial h}{\partial r}\right)^2 G'' + 3\frac{y}{r} G' \frac{\partial h}{\partial r}$$

or

$$\nabla^3 p_o = y\left(\frac{\partial^2 h}{\partial r^2} + \frac{3}{r}\frac{\partial h}{\partial r}\right)G' + y\left(\frac{\partial h}{\partial r}\right)^2 G'' \quad \text{(A-8)}$$

Therefore

$$w = -\frac{\lambda^2}{6} y\left[\left(\frac{\partial^2 h}{\partial r^2} + \frac{3}{r}\frac{\partial h}{\partial r}\right)G' + \left(\frac{\partial h}{\partial r}\right)^2 G''\right] z\,(z+h)^2 \quad \text{(A-9)}$$

APPENDIX B

CIRCULATION OF THE VERTICAL VELOCITY COMPONENT FOR THE CASE

$$h = \frac{1}{1+r^n}$$

$$\left[G = \frac{3\tau_y}{\lambda(2n+1)}\left(\frac{1}{h} + n\right)\right]$$

The components for formula (A-7) are

$$\frac{\partial h}{\partial r} = -nh^2r^{n-1} = -\frac{n}{r}h(1-h) \quad \text{(B-1)}$$

$$\frac{\partial^2 h}{\partial r^2} = \frac{n}{r^2}h(1-h)(1+n-2nh) \quad \text{(B-2)}$$

$$G' = -\frac{3\tau_y}{\lambda(2n+1)}\frac{1}{h^2} \quad \text{(B-3)}$$

$$G'' = \frac{6\tau_y}{\lambda(2n+1)}\ \frac{1}{h^3} \quad \text{(B-4)}$$

Substitution into (A-7) yields

$$w = -\lambda\tau_y \frac{n(n+2)}{2(2n+1)}\frac{y}{r^2}\frac{1-h}{h}z(z+h)^2 \quad \text{(B-5)}$$

APPENDIX C

CALCULATION OF THE VERTICAL VELOCITY COMPONENT FOR THE CASE

$$h = 1 + \frac{\varepsilon}{1+r^n}$$

$$\left[G = \frac{3}{\lambda}\left(\frac{2A}{h} + \frac{B}{2h^2} + D\right)\right]$$

$$\frac{\partial h}{\partial r} = \frac{n}{\varepsilon r}(1-h)(1+\varepsilon-h) \quad \ldots\ldots (C\text{-}1)$$

$$\frac{\partial^2 h}{\partial r^2} = \frac{n}{\varepsilon\ ^2}(1-h)(1+\varepsilon-h)(1+n + \frac{2n}{\varepsilon}(1-h)) \quad \ldots\ldots (C\text{-}2)$$

$$G' = -\frac{3}{\lambda}\frac{1}{h^2}\left(2A + \frac{B}{h}\right) \quad \ldots\ldots (C\text{-}3)$$

$$G'' = 4\frac{3}{\lambda}\frac{1}{h^3}\left(4A + \frac{3B}{h}\right) \quad \ldots\ldots (C\text{-}4)$$

$$w = -\frac{\lambda}{2}\frac{ny}{\varepsilon r^2}\frac{1-h}{h^3}(1+\varepsilon-h)\left\{A\left[4n(1-h)\frac{1+\varepsilon}{\varepsilon} + 2h(n-2)\right]\right.$$

$$\left. + \frac{B}{h}\left[\frac{n}{\varepsilon}(1-h)(3+3\varepsilon-h) + h(n-2)\right]\right\} z(z+h)^2 \quad \ldots\ldots (C\text{-}5)$$